RATIONALIZATION AND IDENTIFICATION OF BINARY GAMES WITH CORRELATED TYPES*

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ABSTRACT. This paper studies the rationalization and identification of binary games where players have correlated private types. Allowing for type correlation is crucial in global games and in models with social interactions as it represents correlated private information and homophily, respectively. Our approach is fully nonparametric in the joint distribution of types and the strategic effects in the payoffs. First, under monotone pure Bayesian Nash Equilibrium strategy, we characterize all the restrictions if any on the distribution of players' choices imposed by the game-theoretic model as well as restrictions associated with two assumptions frequently made in the empirical analysis of discrete games. Namely, we consider exogeneity of payoff shifters relative to private information, and mutual independence of private information given payoff shifters. Second, we study the nonparametric identification of the payoff functions and types distribution. We show that the model with exogenous payoff shifters is fully identified up to a single location–scale normalization under some exclusion restrictions and rank conditions. Third, we discuss partial identification under weaker conditions and multiple equilibria. Lastly, we briefly point out the implications of our results for model testing and estimation.

Keywords: Rationalization, Identification, Discrete Game, Social Interactions, Global Games
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1. INTRODUCTION

Many economic problems are naturally modeled as games of incomplete information (see Morris and Shin, 2003). Over the last decades, such games have been much successful for understanding the strategic interactions among agents in various economic and social situations. A leading example is auctions with e.g. Vickrey (1961), Riley and Samuelson (1981), Milgrom and Weber (1982) for the theoretical side, and Porter (1995), Guerre et al. (2000) and Athey and Haile (2002) for the empirical component. In this paper, we study the identification of static binary games of incomplete information where players have correlated types.[1] We characterize all the restrictions if any imposed by such games on the observables, which are the players' choice probabilities. Following the work by Laffont and Vuong (1996) and Athey and Haile (2007) for auctions, our approach is fully nonparametric.

The empirical analysis of static discrete games is almost thirty years old. The range of applications includes labor force participation (e.g. Bjorn and Vuong, 1984, 1985; Kooreman, 1994; Soetevent and Kooreman, 2007), firms' entry decisions (e.g. Bresnahan and Reiss, 1990, 1991; Berry, 1992; Tamer, 2003; Berry and Tamer, 2006; Jia, 2008; Ciliberto and Tamer, 2009), and social interactions (e.g. Kline, 2015). These papers deal with discrete games under complete information. More recently, discrete games under incomplete information have been used to analyze social interactions by Brock and Durlauf (2001, 2007) and Xu (ming) among others, firm entry and location choices by Seim (2006), timing choices of radio stations commercials by Sweeting (2009), stock market analysts' recommendations by Bajari et al. (2010), capital investment strategies by Aradillas-Lopez (2010) and local grocery markets by Grieco (2014). This list is far from being exhaustive and does not mention the growing literature on estimating dynamic games.

Our paper contributes to this literature in several aspects. First, we focus on monotone pure strategy Bayesian Nash equilibria (BNE) throughout to bridge discrete game modeling with empirical analysis. Monotonicity is a desirable property in many applications for both theoretical and empirical reasons. For instance, White et al. (2014) show that monotone strategies are never worse off than non–monotone strategies in a private value auction

[1] Aradillas-López and Gandhi (2016) study identification and estimation of ordered response games with independent types.

model. On theoretical grounds, Athey (2001) provides seminal results on the existence of a monotone pure strategy BNE whenever a Bayesian game obeys the Spence–Mirlees single–crossing restriction. Relying on the powerful notion of contractibility, Reny (2011) extends Athey's results and related results by McAdams (2003) to give weaker conditions ensuring the existence of a monotone pure strategy BNE. Using Reny's results, we establish the existence of a monotone pure strategy BNE under a weak monotonicity condition on the expected payoff in our setting. This condition is satisfied in most models used in the recent literature. For instance, in empirical IO, it is satisfied when the types are conditionally independent given payoff shifters. In social interaction games, it is also satisfied with strategic complement payoffs and positively regression dependent types. Exceptions that estimate Bayesian games with non–monotone equilibrium include Aradillas-Lopez and Tamer (2008); Xu (2014). In our analysis, the importance of using monotone pure strategy BNEs lies in the fact that we can exploit (weak) monotonicity between observed actions and underlying types to identify nonparametrically the underlying game structure. This opens up the possibility of bringing some theoretical models such as global games (see e.g. Carlsson and Van Damme, 1993; Morris and Shin, 1998) and models with social interactions (see e.g. Galeotti et al., 2010) to nonparametric statistical inference.

Second, we allow players' private information/types to be correlated. In finance and macroeconomics applications of global games (e.g. bank runs, currency crises, and bubbles; see Morris and Shin, 2003), private information are naturally positively correlated. See e.g. Carlsson and Van Damme (1993); Morris and Shin (1998). In oligopoly entry, correlation among types allows us to know "whether entry occurs because of unobserved profitability that is independent of the competition effect" (Berry and Tamer, 2006). In Sociology, correlation among players' types is crucial as it represents the *homophily* phenomenon, which is the principle that people involved in interactions tend to be similar; see e.g. McPherson et al. (2001); Easley and Kleinberg (2010). The recognition of homophily in sociology has a long history: In the writings of Plato, for example, "similarity begets friendship" in his Phaedrus (360 BC). The homophily principle leads to friendship between people with similar demographics (age, race, education, etc) and with positively correlated types (taste, attitudes, etc). The former can be directly observed from the data and has been well documented in empirical sociology. Identifying the latter is more challenging as it is unobserved to the researcher. It is worth pointing out that peer effects and homophily provide two complementary explanations for the common observation that friends tend to behave similarly.[2] Both of them can be separately identified in our framework.

In contrast, mutual independence of private information has been widely assumed in the empirical game literature. See, e.g., Brock and Durlauf (2001); Pesendorfer and Schmidt-Dengler (2003); Seim (2006); Aguirregabiria and Mira (2007); Sweeting (2009); Bajari et al. (2010); Tang (2010); de Paula and Tang (2012); Lewbel and Tang (2015). To our knowledge, the only exceptions are Aradillas-Lopez (2010), Wan and Xu (2014) and Xu (2014). Such an independence of types is a convenient assumption, but imposes strong restrictions such as the mutual independence of players' choices given covariates, a property that is often invalidated by the data.[3] On the other hand, when private information is correlated, the BNE solution concept requires that each player's beliefs about rivals' choices depend on her private information, thereby invalidating the usual two–step identification argument and estimation procedure, see, e.g., Bajari et al. (2010). With such type–dependent beliefs, Wan and Xu (2014) establish some upper/lower bounds for the beliefs in a semiparametric setting with linear–index payoffs. Alternatively, Aradillas-Lopez (2010) adopts a different equilibrium concept related to Aumann (1987), in which each player's equilibrium beliefs do not rely on her private information, but on her actual action.

Third, our analysis is fully nonparametric in the sense that players' payoffs and the joint distribution of players' private information are subject to some mild smoothness conditions only. As far as we know, with the exception of Lewbel and Tang (2015), every paper analyzing empirical discrete games has imposed parametric restrictions on the payoffs and/or the distribution of private information. For instance, Brock and Durlauf (2001); Seim (2006); Sweeting (2009) and Xu (2014) specify both payoffs and the private information distribution parametrically. In a semiparametric context, Aradillas-Lopez (2010); Tang (2010) and Wan and Xu (2014) parameterize players' payoffs, while Bajari et al. (2010) parameterize

[2] In a linear social interaction model, Manski (1993) denotes them as endogenous effects and correlated effects, respectively.

[3] A model with unobserved heterogeneity and independent private information also generates dependence among players' choices conditional on covariates (see e.g. Aguirregabiria and Mira, 2007; Grieco, 2014). See also Section 5.4.

the distribution of private information. Instead, Lewbel and Tang (2015) do not introduce any parameter but impose multiplicative separability in the strategic effect and assume that it is a known function (e.g. sum) of the other players' choices. Our nonparametric baseline discrete game model relaxes such restrictions. We show that such a model imposes essentially no restrictions on the distribution of players' choices. In other words, monotone pure strategy BNEs can explain almost all observed choice probabilities in discrete games.

In view of the preceding result, we consider the identification power and model restrictions associated with two assumptions that are frequently made in the empirical analysis of discrete games. First, we consider the exogeneity of variables shifting players' payoffs relative to players' private information, an assumption that has been frequently imposed in recent empirical work, e.g., Brock and Durlauf (2001); Seim (2006); Sweeting (2009); Aradillas-Lopez (2010); Bajari et al. (2010); de Paula and Tang (2012) and Lewbel and Tang (2015). We show that the resulting model restricts the distribution of players' choices conditional upon payoff shifters and we characterize all those restrictions. Specifically, the exogeneity assumption restricts the joint choice probability to be a monotone function of the corresponding marginal choice probabilities. Given the exogeneity assumption, we show that one can identify the copula function of the types' distribution on an appropriate support. We also show that the equilibrium belief of the player at the margin under a mild support condition. We then characterize the partially identified set of payoffs and the distribution of private information under the exogeneity assumption and the support condition. The partially identified region is unbounded and quite large unless one imposes additional restrictions on the payoffs' functional form.

To achieve point identification, we consider some exclusion restrictions and rank conditions. We show that the players' payoffs are identified up to scale for each fixed value of the exogenous state variables, as well as up to the marginal distributions of players' private information. Moreover, with a single location–scale normalization on the payoff function, we show that both the players' payoffs and distribution of types are fully identified. Our model can be viewed as an extension to a game theoretic setting of traditional threshold–crossing models considered by, e.g., Matzkin (1992). An important difference is that the game setting allows us to exploit exclusion restrictions to achieve nonparametric identification of the distribution of errors. Such restrictions are frequently used in the empirical analysis of discrete games. See, e.g., Aradillas-Lopez (2010); Bajari et al. (2010); Lewbel and Tang (2015) and Wan and Xu (2014).

For completeness, we consider a second assumption, namely the mutual independence of players' private information given payoff shifters. Specifically, we characterize all the restrictions imposed by exogeneity and mutual independence as considered in the empirical game literature. We show that all the restrictions under this pair of assumptions reduce to the conditional independence of players' choices given the payoff shifters. In particular, the restrictions imposed by mutual independence are stronger than those imposed by the exogeneity of payoff shifters and the monotonicity of equilibrium. In other words, any of the latter becomes redundant in explaining players' choices as soon as mutual independence and a single equilibrium are imposed.

The paper is organized as follows. We introduce our baseline model in Section 2. We define and establish the existence of a monotone pure strategy BNE. In Section 3, we study the restrictions imposed by the baseline model. We also derive all the restrictions imposed by the exogeneity and mutual independence assumptions. In Section 4, we establish the nonparametric identification of the model primitives under some support condition, exclusion restrictions and rank conditions. In Section 5, we study the partial identification of the payoffs without exclusion restrictions. We also discuss three related issues: nonparametric estimation, multiple equilibria in the DGP, and unobserved heterogeneity. Section 6 concludes with a brief discussion on testing the model restrictions. An Appendix collects the proofs of our main results. The Appendix also presents a full-fledged example of a binary game with correlated types which we use for verifying the assumptions of the paper from primitive assumptions as well as for illustrating our identification results.

2. Model and Monotone Pure Strategy BNE

We consider a discrete game of incomplete information. There is a finite number of players, indexed by $i = 1, 2, \cdots, I$. Each player simultaneously chooses a binary action $Y_i \in \{0, 1\}$. Let $Y = (Y_1, \cdots, Y_I)$ be an action profile and $\mathcal{A} = \{0, 1\}^I$ be the space of action profiles. Following standard convention, let Y_{-i} and $\mathcal{A}_{-i}$ denote an action profile of all players except i and the corresponding action profile space, respectively. Let $X \in \mathscr{S}_X \subset \mathbb{R}^d$ be a vector of payoff relevant variables, which are publicly observed by all players and also by the researcher.[4] For instance, X can include individual characteristics of the players as well as specific variables for the game environment. For each player i, we further assume that the error term $U_i \in \mathbb{R}$ is her private information, i.e., U_i is observed only by player i, but not by other players. To be consistent with the game theoretic literature, we also call U_i the player i's "type" (see, e.g., Fudenberg and Tirole, 1991). Let $U = (U_1, \cdots, U_I)$ and $F_{U|X}$ be the conditional distribution function of U given X. The conditional distribution $F_{U|X}$ is assumed to be common knowledge.

The payoff of player i is described as follows:

$$\Pi_i(Y, X, U_i) = \begin{cases} \pi_i(Y_{-i}, X) - U_i, & \text{if } Y_i = 1, \\ 0, & \text{if } Y_i = 0, \end{cases}$$

where π_i is a structural function of interest. The zero payoff for action $Y_i = 0$ is a standard payoff normalization in binary response models.[5]

Following the literature on Bayesian games, given the public state variable X, player i's decision rule is a function of her type:

$$Y_i = \delta_i(X, U_i),$$

where $\delta_i : \mathbb{R}^d \times \mathbb{R} \to \{0, 1\}$ maps all the information she knows to a binary decision. For any given strategy profile $\delta = (\delta_1, \cdots, \delta_I)$, let $\sigma^{\delta}_{-i}(a_{-i}|x, u_i)$ be the conditional probability of other players choosing $a_{-i} \in \mathcal{A}_{-i}$ given $X = x$ and $U_i = u_i$, i.e.,

$$\sigma^{\delta}_{-i}(a_{-i}|x, u_i) \equiv \mathbb{P}_{\delta}\left(Y_{-i} = a_{-i}|X = x, U_i = u_i\right) = \mathbb{P}\left[\delta_j(X, U_j) = a_j, \forall j \neq i | X = x, U_i = u_i\right],$$

[4]See Section 5.4 on unobserved heterogeneity when this is not the case. See also Grieco (2014) who analyzes a discrete game that has some payoff relevant variables publicly observed by all players, but not by the researcher.

[5]Here we understand "normalization" from the view of observational equivalence: Suppose the payoffs take the general form:

$$\Pi_i(Y, X, U^*_{i0}, U^*_{i1}) = \begin{cases} \pi^*_{i1}(Y_{-i}, X) - U^*_{i1}, & \text{if } Y_i = 1; \\ \pi^*_{i0}(Y_{-i}, X) - U^*_{i0}, & \text{if } Y_i = 0, \end{cases}$$

where for $y = 0, 1$, U^*_{iy} and π^*_{iy} are action–specific error terms and payoff functions, respectively. It can be shown that this model with our subsequent assumptions is observationally equivalent to the above game with payoff $\pi_i(Y_{-i}, X) = \pi^*_{i1}(Y_{-i}, X) - \pi^*_{i0}(Y_{-i}, X)$ and $U_i = U^*_{i1} - U^*_{i0}$. See also Liu et al. (2012) when $\Pi_i(Y, X, U_i)$ is nonseparable in $U_i \in \mathbb{R}$.

where $\mathbb{P}_\delta$ represents the (conditional) probability measure under the strategy profile δ. The equilibrium concept we adopt is the pure strategy Bayesian Nash equilibrium (BNE). Mixed strategy equilibria are not considered in this paper, since a pure strategy BNE generally exists under weak conditions in our model.

We now characterize the equilibrium solution in the above discrete game. Fix $X = x \in \mathscr{S}_X$. In equilibrium, player i with $U_i = u_i$ chooses action 1 if and only if her expected payoff is greater than zero, i.e.,

$$\delta_i^*(x, u_i) = \mathbf{1}\left[\sum_{a_{-i}} \pi_i(a_{-i}, x) \times \sigma_{-i}^*(a_{-i}|x, u_i) - u_i \geq 0\right], \ \forall\, i, \tag{1}$$

where $\delta^* \equiv (\delta_1^*, \cdots, \delta_I^*)$, as a profile of functions of $u_1, \cdots, u_I$ respectively, denotes the equilibrium strategy profile and $\sigma_{-i}^*(a_{-i}|x, u_i)$ is a shorthand notation for $\sigma_{-i}^{\delta^*}(a_{-i}|x, u_i)$. Note that σ_{-i}^* depends on δ_{-i}^*. Hence, (1) for $i = 1, \cdots, I$ defines a simultaneous equation system in δ^*, referred as "mutual consistency" of players' optimal behaviors. A pure strategy BNE is a fixed point δ^* of such a system, which holds for all $u = (u_1, \ldots, u_I)$ in the support $\mathscr{S}_{U|X=x}$. Ensuring equilibrium existence in Bayesian games is a complex and deep subject in the literature. It is well known that a solution of such an equilibrium generally exists in a broad class of Bayesian games (see, e.g., Vives, 1990).

The key to our approach is to employ a particular equilibrium solution concept of BNE — monotone pure strategy BNEs, which exist under additional weak conditions. Recently, much attention has focused on monotone pure strategy BNEs. The reason is that monotonicity is a natural property and has proven to be powerful in many applications such as auctions, entry, and global games. In our setting, a monotone pure strategy BNE is defined as follows:

Definition 1. *Fix $x \in \mathscr{S}_X$. A pure strategy profile $(\delta_1^*(x,\cdot), \cdots, \delta_I^*(x,\cdot))$ is a monotone pure strategy BNE if $(\delta_1^*(x,\cdot), \cdots, \delta_I^*(x,\cdot))$ is a BNE and $\delta_i^*(x, u_i)$ is (weakly) monotone in u_i for all i.*

Monotone pure strategy BNEs are relatively easier to characterize than ordinary BNEs. Fix $X = x$. In our setting, a monotone pure strategy (m.p.s.) can be explicitly defined as a threshold function (recall that δ_i^* can take only binary values). Formally, in an m.p.s. BNE, player i's equilibrium strategy can be written as $\delta_i^* = \mathbf{1}[u_i \leq u_i^*(x)]$,[6] where $u_i^*(x)$ is the cutoff value that might depend on x. Let $u^*(x) \equiv (u_1^*(x), \cdots, u_I^*(x)) \in \mathbb{R}^I$ be the profile of equilibrium strategy thresholds.

In an m.p.s. BNE, the mutual consistency condition for a BNE solution defined by (1) requires that for each player i,

$$u_i \leq u_i^*(x) \Longleftrightarrow \sum_{a_{-i}} \pi_i(a_{-i}, x) \times \sigma_{-i}^*(a_{-i}|x, u_i) - u_i \geq 0. \tag{2}$$

A simple but key observation is that under certain weak conditions introduced later, (2) implies that player i with the marginal type $u_i^*(x)$ should be indifferent between action 1 and 0, i.e.

$$\textstyle\sum_{a_{-i}} \pi_i(a_{-i}, x) \times \sigma_{-i}^*(a_{-i}|x, u_i^*(x)) - u_i^*(x) = 0. \tag{3}$$

Therefore, the equilibrium strategy can be represented by

$$Y_i = \mathbf{1}\Big[U_i \leq \sum_{a_{-i}} \pi_i(a_{-i}, X) \times \sigma_{-i}^*(a_{-i}|X, u_i^*(X))\Big]. \tag{4}$$

The seminal work on the existence of an m.p.s. BNE in games of incomplete information was first provided by Athey (2001) in both *supermodular* and *logsupermodular* games, and later extended by McAdams (2003) and Reny (2011). Applying Reny (2011) Theorem 4.1, we establish the existence of m.p.s. BNEs in our binary game under some weak regularity assumptions.

Assumption R (Conditional Radon–Nikodym Density)**.** *For every $x \in \mathscr{S}_X$, the conditional distribution of U given $X = x$ is absolutely continuous w.r.t. Lebesgue measure and has a continuous and positive conditional Radon–Nikodym density $f_{U|X}(\cdot|x)$ a.e. over the nonempty interior of its hypercube support $\mathscr{S}_{U|X=x}$.*

Assumption R allows the support of U conditional on $X = x$ to be bounded, namely of the form $\times_{i=1,\dots,I}[\underline{u}_i(x), \overline{u}_i(x)]$ for some finite endpoints $\underline{u}_i(x)$ and $\overline{u}_i(x)$ as frequently used when U_i is i's private information, or unbounded such as when $\mathscr{S}_{U|X=x} = \mathbb{R}^I$ in binary

[6] The left–continuity of strategies considered hereafter is not restrictive given our assumptions below. Note that the payoff function is decreasing in u_i, hence the m.p.s. is also (weakly) decreasing. To simplify, throughout we use "weakly/strictly monotone" to refer to "weakly/strictly decreasing".

response models. As a matter of fact, assumption R can be greatly weakened as shown by Reny (2011) (see Appendix B.2 for more details).

For any strategy profile δ, let $\mathbb{E}_\delta$ denote the (conditional) expectation under the strategy profile δ. Without causing any confusion, we will suppress the subscript δ^* in $\mathbb{E}_{\delta^*}$ (or $\mathbb{P}_{\delta^*}$) when the expectation (or probability) is taken under the equilibrium strategy profile.

Assumption M (Monotone Expected Payoff). *For any weakly m.p.s. profile δ and $x \in \mathscr{S}_X$, the (conditional) expected payoff $\mathbb{E}_\delta\left[\pi_i(Y_{-i}, X)\middle|X = x, U_i = u_i\right] - u_i$ is a weakly monotone function in $u_i \in \mathscr{S}_{U_i|X=x}$.*

Assumption M guarantees that each player's best response is also weakly monotone in type given that all other players adopt weakly m.p.s.. In particular, in a two–player game (i.e., $I = 2$), assumption M is equivalent to the following condition: for any $x \in \mathscr{S}_X$ and $u_{-i} \in \mathbb{R}$, the function $[\pi_i(1, x) - \pi_i(0, x)] \times \mathbb{P}(U_{-i} \leq u_{-i}|X = x, U_i = u_i) - u_i$ is weakly monotone in u_i.

Note that for the existence of m.p.s. BNEs, assumption M is sufficient but not necessary in many cases. It should also be noted that assumption M holds trivially if all U_is are conditionally independent of each other given X. Lemma 6 in Appendix A.2 also provides primitive sufficient conditions for assumption M. Specifically, we assume positive regression dependence across U_is given X and strategic complementarity of players' actions, which are natural restrictions in models with social interactions.

Lemma 1. *Suppose assumptions R and M hold. For any $x \in \mathscr{S}_X$, there exists an m.p.s. BNE. In particular, player i's equilibrium strategy can be written as in* (4).

By Lemma 1, m.p.s. BNEs generally exist in a large class of binary games. As far as we know, with the only exception of Aradillas-Lopez and Tamer (2008) and Xu (2014), every paper analyzing empirical discrete games of incomplete information so far has imposed certain restrictions (i.e. sufficient conditions for assumption M) to guarantee that equilibrium strategies be threshold–crossing.

Monotone pure strategy BNEs are convenient and powerful for empirical analysis. In particular, we can represent each player's equilibrium strategy by a semi–linear–index binary response model (4). Such a representation relates to single-agent binary threshold crossing models studied by e.g. Matzkin (1992), where the 'coefficients' are the player's equilibrium belief about the other players' actions given the threshold signal $u_i^*(x)$. Note, however, that we do not restrict either $\pi_i(a_{-i}, X)$ or $F_{U|X}$ to have a specific functional form. Nevertheless, in Section 4.2 we will show that the equilibrium beliefs σ_{-i}^* in (4) are nonparametrically identified under additional weak conditions.

Though non–monotone strategy BNEs are seldom considered in the literature, it is worth pointing out that this kind of equilibria could exist and sometimes even stands as the only type of equilibria. This could happen when some player is quite sensitive to others' choices and types are highly correlated. We provide a simple example to illustrate.[7]

Example 1. *Let $I = 2$ and $\pi_i = X_i - \beta_i Y_{-i} - U_i$, where (U_1, U_2) conforms to a joint normal distribution with mean zero, unit variances and correlation parameter $\rho \in (-1, 1)$.*[8]

Case 1: Suppose $(X_1, X_2) = (1, 0)$ and $(\beta_1, \beta_2) = (2, 0)$. Then, regardless of the value of ρ, there is always a unique pure strategy BNE: Clearly, player 2 has a dominant strategy which is monotone in u_2: choosing 1 if and only if $u_2 \leq 0$. Thus, player 1's best response must be: choosing 1 if and only if $1 - 2\Phi\big(-\frac{\rho u_1}{\sqrt{1-\rho^2}}\big) - u_1 \geq 0$. Further, it can be shown that player 1's equilibrium strategy is not monotone in u_1 if and only if $\rho \in \big(\sqrt{\frac{\pi}{2+\pi}}, 1\big)$.

Case 2: Suppose $(X_1, X_2) = (1, 1)$ and $(\beta_1, \beta_2) = (2, 2)$. First, note that the m.p.s. profile $\{1(u_1 \leq 0); 1(u_2 \leq 0)\}$ is a BNE as long as $\rho \in \big(-1, \sqrt{\frac{\pi}{2+\pi}}\,\big]$. Moreover, it can be verified that this equilibrium is the unique BNE if and only if $\rho \in \big(-1, \frac{\pi-2}{2+\pi}\,\big]$. When $\rho \in \big(\frac{\pi-2}{2+\pi}, 1\,\big)$, we can find two other equilibria of the game: $\{1(u_1 \leq u^*); 1(u_2 \leq -u^*)\}$ and $\{1(u_1 \leq -u^*); 1(u_2 \leq u^*)\}$ where $u^* > 0$ solves $1 - 2\Phi\big(\sqrt{\frac{1+\rho}{1-\rho}} \cdot u^*\big) + u^* = 0$.

In Case 1, the non–monotone strategy BNE occurs due to the large positive correlation between U_1 and U_2 (relative to β_1), which violates assumption M. On the other hand, for any given value of the structural parameters, the existence of a non–monotone strategy BNE could also depend on the realization of (X_1, X_2).

[7]We thank Steven Stern and Elie Tamer for their comments and suggestions on the following example.

[8]In a similar fully parametric setting, Xu (2014) proposes an inference approach based on first identifying a subset of the covariate space where the game admits a unique m.p.s. BNE.

3. Rationalization

In this section, we study the baseline model defined by assumptions R and M as well as two other models obtained by imposing additional assumptions frequently made in the empirical game literature. Specifically, we characterize all the restrictions imposed on the distribution of observables (Y, X) by each of these models.

We say that a conditional distribution $F_{Y|X}$ is rationalized by a model if and only if it satisfies all the restrictions of the model. Equivalently, $F_{Y|X}$ is rationalized by the model if and only if there is a structure (not necessarily unique) in the model that generates such a distribution. In particular, rationalization logically precedes identification as the latter, which is addressed in Section 4, makes sense only if the observed distribution can be rationalized by the model under consideration.

An assumption frequently made in the literature, e.g., Brock and Durlauf (2001) and Bajari et al. (2010), is the exogeneity of the observed state variables X relative to private information U.

Assumption E (Exogeneity)**.** *X and U are independent of each other.*[9]

Another assumption called as mutual independence has been also widely used in the literature. For examples, see an extensive list of references in two recent surveys: Bajari et al. (2010) and de Paula (2013). Such an independence of types is a convenient theoretical assumption, which means player i's private information is uninformative about other players' types given X.

Assumption I (Mutual Independence)**.** *$U_1, \cdots, U_I$ are mutually independent conditional on X.*

Let $S \equiv [\pi; F_{U|X}]$, where $\pi = (\pi_1, \cdots, \pi_I)$. We now consider the following models:

$$\mathcal{M}_1 \equiv \big\{S : \text{Assumptions R and M hold and a single m.p.s. BNE is played}\big\},$$

$$\mathcal{M}_2 \equiv \{S \in \mathcal{M}_1 : \text{ Assumption E holds}\},$$

$$\mathcal{M}_3 \equiv \{S \in \mathcal{M}_2 : \text{ Assumption I holds}\}.$$

[9] Our results can be easily extended to the weaker assumption that X and U are independent from each other conditional on W, where W are other observed payoff relevant variables.

Clearly, $\mathcal{M}_1 \supsetneq \mathcal{M}_2 \supsetneq \mathcal{M}_3$.

The last requirement in $\mathcal{M}_1$ is not restrictive when the game has a unique equilibrium which has to be an m.p.s. BNE under assumptions R and M. In the global game literature, for example, uniqueness of m.p.s. BNE is achieved as the information noise gets small. See e.g. Carlsson and Van Damme (1993) and Morris and Shin (1998). In social interactions, uniqueness of m.p.s. BNE has also been established in e.g. Brock and Durlauf (2001) and Xu (ming). When there exist multiple equilibria, we follow part of the literature by assuming that the same equilibrium is played in the DGP for any given x. See e.g. Aguirregabiria and Nevo (2013) for a survey. Such an assumption is realistic if the equilibrium selection rule is actually governed by some game invariant factors, like culture, social norm, etc. See, e.g., de Paula (2013) for a detailed discussion. Relaxing such a requirement has been addressed in recent work and will be discussed in Section 5.3.

We introduce some key notation for the following analysis. For any structure $S \in \mathcal{M}_1$, let $\alpha_i(x) \equiv F_{U_i|X}(u_i^*(x)|x)$. By monotonicity of the equilibrium strategy, we have $\alpha_i(x) = \mathbb{E}(Y_i|X = x)$, i.e. $\alpha_i(x)$ is player i's (marginal) probability of choosing action 1 given $X = x$. Moreover, for each $p = 2, \cdots, I$, and $1 \leq i_1 < \cdots < i_p \leq I$, let $C_{U_{i_1},\cdots,U_{i_p}|X}$ be the conditional copula function of $(U_{i_1}, \cdots, U_{i_p})$ given X, i.e., for any $(\alpha_{i_1}, \cdots, \alpha_{i_p}) \in [0,1]^p$ and $x \in \mathscr{S}_X$,

$$C_{U_{i_1},\cdots,U_{i_p}|X}(\alpha_{i_1}, \cdots, \alpha_{i_p}|x) \equiv F_{U_{i_1},\cdots,U_{i_p}|X}\left(F^{-1}_{U_{i_1}|X}(\alpha_{i_1}|x), \cdots, F^{-1}_{U_{i_p}|X}(\alpha_{i_p}|x)\Big|x\right).$$

The next proposition characterizes the collection of distributions of Y given X that can be *rationalized* by $\mathcal{M}_1$.

Proposition 1. *A conditional distribution $F_{Y|X}$ is rationalized by $\mathcal{M}_1$ if and only if for all $x \in \mathscr{S}_X$ and $a \in \mathcal{A}$, $\mathbb{P}(Y = a|X = x) = 0$ implies that $\mathbb{P}(Y_i = a_i|X = x) = 0$ for some i.*

By Proposition 1, $\mathcal{M}_1$ rationalizes all distributions of Y given X that belong to the interior of the $2^I - 1$ dimensional simplex, i.e. distributions with strictly positive choice probabilities, since the condition in Proposition 1 is void for such distributions. Specifically, the distributions that cannot be rationalized by $\mathcal{M}_1$ must have $\mathbb{P}(Y = a|X = x) = 0$ for some $a \in \mathcal{A}$, i.e., distributions for which there are "structural zeros." In other words, our baseline model $\mathcal{M}_1$ imposes no essential restrictions on the distribution of observables. The distributions that cannot be rationalized by $\mathcal{M}_1$ arise because of assumption R. As noted earlier, one can replace assumption R by Reny (2011)'s weaker conditions, in which case any distribution for Y given X can be rationalized. See Lemma 7 in Appendix B.2.

We now characterize all the restrictions imposed on $F_{Y|X}$ by model $\mathcal{M}_2$. These additional restrictions come from assumption E.

Proposition 2. *A conditional distribution $F_{Y|X}$ rationalized by $\mathcal{M}_1$ is also rationalized by $\mathcal{M}_2$ if and only if for each $p = 2, \cdots, I$ and $1 \le i_1 < \cdots < i_p \le I$,*

R1: $\mathbb{E}\big(\prod_{j=1}^{p} Y_{i_j} \big| X\big) = \mathbb{E}\big(\prod_{j=1}^{p} Y_{i_j} \big| \alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)\big)$.

R2: $\mathbb{E}\big(\prod_{j=1}^{p} Y_{i_j} | \alpha_{i_1}(X) = \cdot, \cdots, \alpha_{i_p}(X) = \cdot\big)$ *is strictly increasing on* $\mathscr{S}_{\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)}$ *except at values for which some coordinates are zero.*

R3: $\mathbb{E}\big(\prod_{j=1}^{p} Y_{i_j} | \alpha_{i_1}(X) = \cdot, \cdots, \alpha_{i_p}(X) = \cdot\big)$ *is continuously differentiable on* $\mathscr{S}_{\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)}$.

In Proposition 2, the most stringent restriction is R1, which requires that the joint choice probability depend on X only through the corresponding marginal choice probabilities. Under restrictions R1 and R2, the condition $\alpha(x) \ge \alpha(x')$ implies that $\mathbb{P}(Y_{i_1} = 1, \cdots, Y_{i_p} = 1 | X = x) \ge \mathbb{P}(Y_{i_1} = 1, \cdots, Y_{i_p} = 1 | X = x')$ for all tuples $\{i_1, \cdots, i_p\}$. Moreover, note that $\alpha_i(x)$ is identified by $\alpha_i(x) = \mathbb{E}(Y_i | X = x)$. Therefore, all the restrictions R1–R3 are testable in principle.[10] This is discussed further in the Conclusion.

For completeness, we also study the restrictions on observables imposed by $\mathcal{M}_3$, which makes the additional assumption I. It should be noted that assumption M is satisfied when assumption I holds. In other words,

$$\mathcal{M}_3 = \{S : \text{Assumptions R, E, I hold and a single m.p.s. BNE is played}\}.$$

In the literature, several special cases of $\mathcal{M}_3$ have been considered under some parametric assumptions, see, e.g., Bajari et al. (2010).

[10] If X is discrete, then R3 becomes irrelevant.

Proposition 3. *A conditional distribution $F_{Y|X}$ can be rationalized by $\mathcal{M}_3$ if and only if $Y_1, \cdots, Y_I$ are conditionally independent given X, i.e. for each $p = 2, \cdots, I$ and $1 \leq i_1 < \cdots < i_p \leq I$, $\mathbb{E}\big(\prod_{j=1}^{p} Y_{i_j}|X\big) = \prod_{j=1}^{p} \alpha_{i_j}(X)$.*

It is worth pointing out that Propositions 1 to 3 exhaust all possible testable restrictions as they provide necessary *and* sufficient conditions for rationalizing $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, respectively. Moreover, their proofs are constructive. Specifically, we construct an I–single–agent decision structure that rationalizes the given distributions satisfying the corresponding restrictions. This is summarized by the following corollary. For $k = 1, 2, 3$, let

$$\mathcal{M}_k^s = \{S \in \mathcal{M}_k : \ \pi_i(a'_{-i}, x) = \pi_i(a_{-i}, x), \forall a'_{-i}, a_{-i} \in \mathcal{A}_{-i}, x \in \mathscr{S}_X \text{ and } i = 1, \cdots, I\}.$$

Corollary 1. *For $k = 1, 2, 3$, $\mathcal{M}_k$ is observationally equivalent to $\mathcal{M}_k^s$.*

Hence, it is evident that without additional model restrictions beyond the assumptions of $\mathcal{M}_1$ $\mathcal{M}_2$, or $\mathcal{M}_3$, a discrete Bayesian game model with strict interactions cannot be empirically distinguished from an alternative model with I–single–agent decisions. In contrast, with exclusion restrictions, Section 4 shows that these two classes of models can be distinguished from each other.

It should also be noted that the conditional independence restriction in Proposition 3 implies conditions R1–R3 in Proposition 2, as well as the necessary and sufficient condition in Proposition 1. The conditional independence of players' choices given payoffs shifters characterizing $\mathcal{M}_3$ suggests that we can replace $\mathcal{M}_3$ in Proposition 3 with

$$\mathcal{M}'_3 \equiv \{S : \text{ Assumption I holds and a single BNE is played}\},$$

since $\mathcal{M}'_3$ also implies the conditional independence restriction. In particular, $\mathcal{M}'_3$ does not require the monotonicity of the BNE. Because $\mathcal{M}_3 \subset \mathcal{M}'_3$, we have the following corollary.

Corollary 2. *Model $\mathcal{M}_3$ imposes the same restrictions on the distribution of observables as $\mathcal{M}'_3$, i.e., both models are observationally equivalent.*

This is a surprising result: Assumptions R, M and more importantly exogeneity of the payoff shifters (assumption E) become redundant in terms of restrictions on the observables, as soon as mutual independence of types conditional on X (assumption I) and a single BNE condition are imposed on the baseline model. Moreover, if we are willing to maintain assumption I, then Proposition 3 and Corollary 2 gives us a test of a single equilibrium being played, as rejecting the conditional independence of the players' choices given X indicates the presence of multiple equilibria. This extends a related result in terms of correlation obtained by de Paula and Tang (2012) in a partial–linear setting.

4. Nonparametric Identification

In this section we study the nonparametric identification of the baseline model $\mathcal{M}_1$, and its special cases $\mathcal{M}_2$ and $\mathcal{M}_3$. The recent literature has focused on the parametric or semiparametric identification of structures in $\mathcal{M}_3$, see, e.g., Brock and Durlauf (2001); Seim (2006); Sweeting (2009); Bajari et al. (2010), and Tang (2010). As far as we know, Lewbel and Tang (2015) is the only paper that studies the nonparametric identification of a submodel of $\mathcal{M}_3$ obtained through additional restrictions on the functional form of payoffs.

In our context, identification of each model is equivalent to identification of the payoffs π_i, the marginal distribution function $F_{U_i|X}$ and the copula function $C_{U|X}$ of players' types. Let $Q_{U_i|X}$ be the quantile function of $F_{U_i|X}$. Because the quantile function is the inverse of the CDF, i.e. $Q_{U_i|X} = F_{U_i|X}^{-1}$, identification reduces to that of the triple $\left[\pi; \{Q_{U_i|X}\}_{i=1}^{I}; C_{U|X}\right]$. In contrast to single-agent binary threshold crossing models, we do not require any of such primitives to be parameterized.

We first show that $\mathcal{M}_1$ is not identified in general. For $\mathcal{M}_2$, we first establish the identification of $C_{U|X}$ and the equilibrium beliefs $\sigma_{-i}^*(\cdot|x, u_i^*(x))$ under an additional support condition. The identification of the copula function $C_{U|X}$ is of particular interest in social interactions, since it represents (unobserved) homophily among friends. Under some exclusion restrictions and rank conditions, we then establish the identification of the payoff functions π and the quantile functions $\{Q_{U_i|X}\}_{i=1}^{I}$ up to a single location–and–scale normalization on the payoffs. Regarding $\mathcal{M}_3$, its identification requires slightly weaker support restrictions than those for $\mathcal{M}_2$, though the differences are not essential.

4.1. Nonidentification of $\mathcal{M}_1$.

We begin with the most general model $\mathcal{M}_1$.

Proposition 4. *$\mathcal{M}_1$ is not identified nonparametrically.*

The proof is trivial. It follows directly from the observational equivalence between any structure S in $\mathcal{M}_1$ and a collection of I–single-agent binary responses models: Let $\tilde{S} \equiv \big(\tilde{\pi}; \tilde{F}_{U|X}\big)$, in which $\tilde{\pi}_i(\cdot, x) \equiv \tilde{u}_i(x)$, where $\tilde{u}_i(x)$ is arbitrarily chosen, and $\tilde{F}_{U|X}$ satisfies assumption R with $\tilde{F}_{U_{i_1},\cdots,U_{i_p}|X}(\tilde{u}_{i_1}(x), \cdots, \tilde{u}_{i_p}(x)|x) = F_{U_{i_1},\cdots,U_{i_p}|X}(u^*_{i_1}(x), \cdots, u^*_{i_p}(x)|x)$ for all $x \in \mathscr{S}_X$ and all tuples $\{i_1, \cdots, i_p\}$. Thus, $\tilde{S}$ and S are observationally equivalent thereby establishing the non–identification of $\mathcal{M}_1$.[11]

Next, we turn to the identification of $\mathcal{M}_2$ and its sub–model $\mathcal{M}_3$. First note that we maintain assumption E in both models, i.e. that X and U are independent of each other; See Footnote 9 for a weaker assumption. It follows that $Q_{U_i|X} = Q_{U_i}$ and $C_{U|X} = C_U$. Thus, identification of these models reduces to that of the triple $\big[\pi; \{Q_{U_i}\}_{i=1}^I; C_U\big]$.

4.2. **Identification of $\mathcal{M}_2$.** Let $\alpha(x) \equiv (\alpha_1(x), \cdots, \alpha_I(x))$ be a profile of the marginal choice probabilities. Note that $\alpha_i(x)$ is identified by $\mathbb{E}(Y_i|X = x)$. Under assumption E, the copula function C_U is nonparametrically identified on an appropriate domain, namely, the extended support of $\alpha(X)$ defined as $\mathscr{S}^e_{\alpha(X)} \equiv \{\alpha : \alpha_j = 0 \text{ for some } j\} \bigcup \big\{\alpha : (\alpha_{i_1}, \cdots, \alpha_{i_p}) \in \mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}; \text{other } \alpha_{i_j} = 1\big\}$. We have $C_U(\alpha) = 0$ if $\alpha_j = 0$ for some j; otherwise,[12]

$$
\begin{aligned}
C_U(\alpha) &= \mathbb{P}\big\{U_1 \leq Q_{U_1}(\alpha_1), \cdots, U_I \leq Q_{U_I}(\alpha_I)\big\} \\
&= \mathbb{P}\Big[U_j \leq Q_{U_j}(\alpha_j),\ \forall j \in \{i : \alpha_i \neq 1\}\Big] = \mathbb{E}\Big[\prod_{i=1}^{I} Y_i \big| \alpha_j(X) = \alpha_j, \forall j \in \{i : \alpha_i \neq 1\}\Big]. \quad (5)
\end{aligned}
$$

Key among those conditions for the nonparametric identification of C_U is the assumption that a single m.p.s. BNE is played in the DGP. Such a restriction implies that conditional on $\alpha_j(X) = \alpha_j$, the event $U_j \leq Q_{U_j}(\alpha_j)$ is equivalent to $Y_j = 1$.

As mentioned above, the equilibrium belief $\sigma^*_{-i}(\cdot|x, u^*_i(x))$ can also be nonparametrically identified, for which we need a support condition on $\alpha(X)$.

Assumption SC (Support Condition)**.** *The support $\mathscr{S}_{\alpha(X)}$ is the closure of a nonempty open set.*

[11] Even if one imposes the identifying restrictions (namely the exclusion restriction, support condition and rank condition) introduced later, $\mathcal{M}_1$ is still not identified by a similar argument.

[12] In (5), it is understood that $C_U(\alpha) = 1$ if $\alpha = (1, \cdots, 1)$.

Assumption SC implies that the dimension of the interior $\mathscr{S}^{\circ}_{\alpha(X)}$ of $\mathscr{S}_{\alpha(X)}$ is I. Therefore, we can take derivatives in all directions of an arbitrary smooth function defined on $\mathscr{S}^{\circ}_{\alpha(X)}$.[13] Moreover, given the identification of $\alpha(x)$, assumption SC is verifiable. It is worth pointing out that Assumption SC does not necessarily require the dimension of X to be larger than or equal to the number of players.

Lemma 2. *Let $S \in \mathcal{M}_2$. Suppose assumption SC holds. Fix $x \in \mathscr{S}_X$. Then the equilibrium beliefs $\sigma^*_{-i}(\cdot|x, u^*_i(x))$ are identified. Namely, for all $a_{-i} \in \mathcal{A}_{-i}$,*

$$\sigma^*_{-i}(a_{-i}|x, u^*_i(x)) = \left.\frac{\partial \mathbb{P}\left(Y_i = 1; Y_{-i} = a_{-i}|\alpha(X) = \alpha\right)}{\partial \alpha_i}\right|_{\alpha=\alpha(x)}. \tag{6}$$

Note that, under assumption I, the probability $\mathbb{P}\left(Y_i = 1; Y_{-i} = a_{-i}|\alpha(X)\right) = \alpha_i(X) \times \prod_{j\neq i} \alpha_j^{a_j}(X)[1-\alpha_j(X)]^{1-a_j}$ becomes a (known) linear function in $\alpha_i(X)$. Thus, we have $\sigma^*_{-i}(a_{-i}|x, u^*_i(x)) = \prod_{j\neq i} \alpha_j^{a_j}(X)[1-\alpha_j(X)]^{1-a_j}$, thereby identifying trivially the equilibrium beliefs without assumption SC, see, e.g., Bajari et al. (2010).

To illustrate the intuition of Lemma 2, we use a two–player game.

Example 2. *Let $S \in \mathcal{M}_2$ and $I = 2$. Note that for any $\alpha \in [0,1]^2$, we have*

$$\mathbb{P}(Y_1 = 1, Y_2 = 1|\alpha(X) = \alpha) = \mathbb{P}\left[U_1 \le Q_{U_1}(\alpha_1), U_2 \le Q_{U_2}(\alpha_2)\right] = C_U(\alpha_1, \alpha_2),$$

where the first equality follows from $\alpha_i(X) = \alpha_i$ being equivalent to $u^*_i(X) = Q_{U_i}(\alpha_i)$ from the independence of U and X. Further, we have

$$\frac{\partial C_U(\alpha_1, \alpha_2)}{\partial \alpha_i} = \mathbb{P}\left(U_{-i} \le Q_{U_{-i}}(\alpha_{-i})|U_i = Q_{U_i}(\alpha_i)\right), \tag{7}$$

see, e.g., Darsow et al. (1992). Because $Q_{U_i}(\alpha_i(x)) = u^*_i(x)$, it follows that

$$\begin{aligned}\left.\frac{\partial \mathbb{P}(Y_1 = 1, Y_2 = 1|\alpha(X) = \alpha)}{\partial \alpha_i}\right|_{\alpha=\alpha(x)} &= \mathbb{P}\left(U_{-i} \le u^*_{-i}(x)|U_i = u^*_i(x)\right) \\ &= \mathbb{P}\left(Y_{-i} = 1|X = x, U_i = u^*_i(x)\right) = \sigma^*_{-i}(1|x, u^*_i(x)).\end{aligned}$$

[13] As a matter of fact, for the boundary points, we can take directional derivatives as well.

Similarly, we have

$$\frac{\partial \mathbb{P}(Y_i = 1, Y_{-i} = 0 | \alpha(X) = \alpha)}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)} = \sigma^*_{-i}(0|x, u^*_i(x)).$$

Equation (7) is related to the treatment effect literature, e.g., Heckman and Vytlacil (1999, 2005), Carneiro and Lee (2009) and Jun et al. (2011). Taking derivative with respect to the propensity score identifies the conditional quantile (or conditional expectation) of the treatment effect at the margin. Lemma 2 extends this result to the multivariate case by using the law of iterated expectation.

We now discuss the identification of π and Q_{U_i}. Fix $X = x$ such that $\alpha_i(x) \in (0, 1)$. Because $u^*_i(x) = Q_{U_i}(\alpha_i(x))$, we represent the equilibrium condition (3) by

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x) \times \sigma^*_{-i}(a_{-i}|x, u^*_i(x)) - Q_{U_i}(\alpha_i(x)) = 0, \tag{8}$$

where σ^*_{-i} is known by Lemma 2. Next, we will exploit (8) for the identification of the payoffs π_i. The idea is to vary $\sigma^*_{-i}(\cdot|x, u^*_i(x))$ while keeping $\pi_i(\cdot, x)$ fixed, for which we need the following exclusion restriction.

Assumption ER (Exclusion Restriction)**.** *Let $X = (X_1, \cdots, X_I)$. For all i, a_{-i} and x, we have $\pi_i(a_{-i}, x) = \pi_i(a_{-i}, x_i)$.*[14]

In the context of discrete games, the identification power of exclusion restrictions was first demonstrated in Pesendorfer and Schmidt-Dengler (2003), Tamer (2003), and was used by Bajari et al. (2010) in a semiparametric setting. For instance, in empirical IO, some cost shifters are included in the payoff of firm i but not in firm j's, and vice versa.

Under assumption ER, (8) implies that

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \Big\{ \sigma^*_{-i}(a_{-i}|x, u^*_i(x)) - \mathbb{E}\left[\sigma^*_{-i}\big(a_{-i}|X, u^*_i(X)\big)|X_i = x_i, \alpha_i(X) = \alpha_i(x)\right] \Big\} = 0. \tag{9}$$

For notational simplicity, we denote the random vector $\sigma^*_{-i}(\cdot|X, u^*_i(X))$ as $\Sigma^*_{-i}(X)$, a column vector of dimension 2^{I-1}. Let $\overline{\Sigma}^*_{-i}(X) \equiv \Sigma^*_{-i}(X) - \mathbb{E}\left[\Sigma^*_{-i}(X)|X_i, \alpha_i(X)\right]$ and $\mathcal{R}_i(x_i) =$

[14] As a matter of fact, X_is can have some common variables due to homophily. In this case, our results hold by conditioning on those common variables.

$\mathbb{E}\left[\overline{\Sigma}^*_{-i}(X)\overline{\Sigma}^*_{-i}(X)^\top | X_i = x_i\right]$. Given Lemma 2, we treat $\Sigma^*_{-i}(X)$ and $\overline{\Sigma}^*_{-i}(X)$ as observables hereafter. Note that $\iota'\Sigma^*_{-i}(X) = 1$ a.s., where $\iota \equiv (1, \cdots, 1)' \in \mathbb{R}^{2^{I-1}}$. It follows that $\iota'\overline{\Sigma}^*_{-i}(X) = 0$. Thus, $\overline{\Sigma}^*_{-i}(X)$ consists of a vector of linearly dependent variables. Indeed, the largest possible rank of the matrix $\mathcal{R}_i(x_i)$ is $2^{I-1} - 1$. In the next proposition, we give identification results for features of $\mathcal{M}_2$.

Lemma 3. *Suppose $S \in \mathcal{M}_2$ and assumptions SC and ER hold. Fix $x_i \in \mathscr{S}_{X_i}$ such that $\mathscr{S}_{\alpha_i(X)|X_i=x_i} \bigcap (0,1) \neq \emptyset$. If the rank of $\mathcal{R}_i(x_i)$ is $2^{I-1} - 1$, then $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ must be a singleton $\{\alpha_i^\dagger\}$ and $\pi_i(\cdot, x_i)$ is identified up to the $\alpha_i^\dagger$–quantile of F_{U_i}, i.e. $\pi_i(\cdot, x_i) = Q_{U_i}(\alpha_i^\dagger)$. If the rank of $\mathcal{R}_i(x_i)$ is $2^{I-1} - 2$, then $\pi_i(\cdot, x_i)$ is identified up to location and scale that depend on x_i, or equivalently, $\pi_i(\cdot, x_i) - \pi_i(a^0_{-i}, x_i)$ is identified up to scale for arbitrary $a^0_{-i} \in \mathcal{A}_{-i}$.*

Lemma 3 shows that fixing x_i, the payoff function $\pi_i(\cdot, x_i)$ is identified as a constant, or identified up to location and scale, where the scale could be either positive or negative. In particular, if $\mathcal{R}_i(x_i)$ has the largest rank $2^{I-1} - 1$, there are no strategic effects. Thus, one can test the lack of strategic interactions by testing such a rank condition.

For example, let $I = 3$. Fix $X_i = x_i$. Then, (9) contains four unknown coefficients $\pi_i((0,0), x_i), \pi_i((0,1), x_i), \pi_i((1,0), x_i)$ and $\pi_i((1,1), x_i)$. Note that $\Sigma^*_{-i}(X) \equiv \sigma^*_{-i}(\cdot|X, u_i^*(X))$ is a random vector distributed on a simplex of dimension three. In addition, controlling for $\alpha_i(X)$, $\mathbb{E}\left[\Sigma^*_{-i}(X)|X_i, \alpha_i(X)\right]$ is a (random) vector in the same simplex. Thus, $\overline{\Sigma}^*_{-i}(X)$ belongs to the the re-centered simplex, which is obtained by shifting the three–dimensional simplex to be centered at $(0,0,0,0)$. Because $\overline{\Sigma}^*_{-i}(X)$ satisfies (9), then $\overline{\Sigma}^*_{-i}(X)$ is distributed on the intersection of the three–dimensional hyperplane defined by (9) and the re-centered three–dimensional simplex. The intersection is a two–dimensional triangle unless the hyperplane contains the re-centered simplex. In the former case, the rank $\mathcal{R}_i(x_i) = \mathbb{E}\left[\overline{\Sigma}^*_{-i}(X)\overline{\Sigma}^*_{-i}(X)^\top | X_i = x_i\right] \leq 2$, while the latter case implies $\mathcal{R}_i(x_i) \leq 3$. Thus, when $\mathcal{R}_i(x_i) = 3$, only the latter case applies. Hence, the coefficients in (9) are determined proportionally by the re-centered probability–mass simplex: $\pi_i((0,0), x_i) = \pi_i((0,1), x_i) = \pi_i((1,0), x_i) = \pi_i((1,1), x_i)$, i.e., there are no strategic effects on i.

Under an additional assumption (i.e. assumption V below), we identify the existence of strategic effects. In addition, we can identify the sign of $\pi_i(a_{-i}, x_i) - \pi_i(a'_{-i}, x_i)$. When players' private signals are independent, de Paula and Tang (2012) develop a special approach for nonparametrically identifying the signs of the strategic effects by exploiting the identification power of multiple equilibria. In contrast, our approach relies on assumptions E and ER while being applicable when there is only one (m.p.s.) equilibrium.

Assumption V (Variations in Marginal Choice Probabilities)**.** *Fix $x_i \in \mathscr{S}_{X_i}$. There exist $\alpha, \alpha' \in \mathscr{S}_{\alpha(X)|X_i=x_i}$ such that $0 < \alpha_i \neq \alpha'_i < 1$ and $(\alpha_i, \alpha'_{-i}) \in \mathscr{S}_{\alpha(X)}$.*

Proposition 5. *Suppose $S \in \mathcal{M}_2$. Fix $x_i \in \mathscr{S}_{X_i}$. Then $\pi_i(\cdot, x_i)$ varies on $\mathcal{A}_{-i}$ if $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ is not a singleton. Moreover, suppose assumptions SC, ER and V hold. If the rank of $\mathcal{R}_i(x_i)$ is $2^{I-1}-2$, then the sign of $\pi_i(a_{-i}, x_i) - \pi_i(a^0_{-i}, x_i)$ is identified for each $a_{-i} \in \mathcal{A}_{-i}$.*

When assumption V holds, the rank condition in Proposition 5 requires that X_{-i} contain at least one continuous random variable such that, conditional on $X_i = x_i$ and $\alpha_i(X) = \alpha_i$, there are sufficient variations in $\sigma^*_{-i}(\cdot|X, u^*_i(X))$ by varying X_{-i}. Such a rank condition is related to Pesendorfer and Schmidt-Dengler (2003) and Bajari et al. (2010) under semiparametric settings.

To identify the payoffs up to a single location and scale, we introduce a normalization. Similar to Matzkin (2003), our normalization is imposed on the payoff functions at some $x^*_i \in \mathscr{S}_{X_i}$.

Assumption N (Payoff Normalization)**.** *We set $\pi_i\left(a^0_{-i}, x^*_i\right) = 0$ and $\|\pi_i(\cdot, x^*_i)\| = 1$ for some $x^*_i \in \mathscr{S}_{X_i}$ satisfying (i) assumption V and (ii) the rank of $\mathcal{R}_i(x^*_i)$ is $2^{I-1}-2$.*[15]

Let $S \in \mathcal{M}_2$. Suppose that assumptions SC, ER and N hold. By Lemma 3, the payoffs $\pi_i(\cdot, x^*_i)$ is point identified. By (8), Q_{U_i} is identified on the support $\mathscr{S}_{\alpha_i(X)|X_i=x^*_i} \bigcap (0,1)$. Further, for each $x_i \in \mathscr{S}_{X_i}$, suppose that the rank of $\mathcal{R}_i(x_i)$ equals $2^{I-1}-2$, and that $\mathscr{S}_{\alpha_i(X)|X_i=x_i} \bigcap \mathscr{S}_{\alpha_i(X)|X_i=x^*_i} \bigcap (0,1)$ contains two elements $\alpha_i, \alpha'_i \in (0,1)$. By Lemma 3, $\pi_i(\cdot, x_i)$ is identified up to location and scale. Note that the quantiles $Q_{U_i}(\alpha_i)$ and $Q_{U_i}(\alpha'_i)$ are known since $\alpha_i, \alpha'_i \in \mathscr{S}_{\alpha_i(X)|X_i=x^*_i} \bigcap (0,1)$. Therefore, we can determine the location and scale of

[15] W.l.o.g., we set $a^0_{-i} = (0, \cdots, 0) \in \mathcal{A}_{-i}$. Note that $\|\pi_i(\cdot, x^*_i) - \pi_i(a^0_{-i}, x^*_i)\| \neq 0$ because of Assumption V.

$\pi_i(\cdot, x_i)$ from the following two equations

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \times \mathbb{E}\left[\sigma^*_{-i}(a_{-i}|X, u^*_i(X))|X_i = x_i, \alpha_i(X) = \alpha_i\right] = Q_{U_i}(\alpha_i);$$
$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \times \mathbb{E}\left[\sigma^*_{-i}(a_{-i}|X, u^*_i(X))|X_i = x_i, \alpha_i(X) = \alpha'_i\right] = Q_{U_i}(\alpha'_i).$$

Moreover, we can identify Q_{U_i} on the support $\mathscr{S}_{\alpha_i(X)|X_i \in \{x^*_i, x_i\}} \bigcap (0,1)$. Repeating such an argument, we can show that $\pi_i(\cdot, x_i)$ is point identified for all x_is in a collection, denoted as $\mathbb{C}^\infty_i$, while Q_{U_i} is identified on the support $\mathscr{S}_{\alpha_i(X)|X_i \in \mathbb{C}^\infty_i} \bigcap (0,1)$.

Definition 2. *Let the subset $\mathbb{C}^\infty_i$ in $\mathscr{S}_{X_i}$ be defined by the following iterative scheme. Let $\mathbb{C}^0_i = \{x^*_i\}$. Then, for all $t \geq 0$, $\mathbb{C}^{t+1}_i$ consists of all elements $x_i \in \mathscr{S}_{X_i}$ such that at least one of the following conditions is satisfied: (i) $x_i \in \mathbb{C}^t_i$; (ii) $\mathcal{R}_i(x_i)$ has rank $2^{I-1} - 2$ and there exists an $x'_i \in \mathbb{C}^t_i$ such that $\mathscr{S}_{\alpha_i(X)|X_i = x_i} \bigcap \mathscr{S}_{\alpha_i(X)|X_i = x'_i} \bigcap (0,1)$ contains at least two different elements; and (iii) $\mathcal{R}_i(x_i)$ has rank $2^{I-1} - 1$ and there exists an $x'_i \in \mathbb{C}^t_i$ such that $\mathscr{S}_{\alpha_i(X)|X_i = x_i} \subseteq \mathscr{S}_{\alpha_i(X)|X_i = x'_i} \bigcap (0,1)$.*

In view of Lemma 3, condition (ii) in Definition 2 corresponds to the case where there are strategic effects. This case is the key to effectively expand the collection of x_is in an iterative manner by enlarging $\mathscr{S}_{\alpha_i(X)|X_i \in \mathbb{C}^t_i}$ to $\mathscr{S}_{\alpha_i(X)|X_i \in \mathbb{C}^{t+1}_i}$. Note that to exploit condition (ii), we implicitly assume that X_{-i} contains at least one continuous random variable.

Proposition 6. *Let $S \in \mathcal{M}_2$. Suppose assumptions SC, ER and N hold. Then π_i and Q_{U_i} are point identified on the support $\mathcal{A}_{-i} \times \mathbb{C}^\infty_i$ and $\mathscr{S}_{\alpha_i(X)|X_i \in \mathbb{C}^\infty_i} \bigcap (0,1)$, respectively.*

It is interesting to note that our identification argument does not apply to a nonparametric single–agent binary response model, see e.g. Matzkin (1992).[16] This is because the support $\mathscr{S}_{\alpha_i(X)|X_i = x^*_i}$ is a singleton in a single–agent binary response model, i.e., we are always in the case of condition (iii) in Definition 2. In contrast, with interactions and exclusion restrictions, we can exploit variations of X_{-i} while controlling for X_i to identify a set of quantiles of F_{U_i}.

Note that $\{\mathbb{C}^t_i : t \geq 1\}$ is an expanding sequence on the support $\mathscr{S}_{X_i}$, which ensures that the limit $\mathbb{C}^\infty_i$ is well defined (and may not be bounded if $\mathscr{S}_{X_i}$ is unbounded). The domain and size of $\mathbb{C}^\infty_i$ depend on the choice of x^*_i as well as the variation of $\mathscr{S}_{\alpha_i(X)|X_i = x_i}$ across

[16] In single–agent binary response models, Matzkin (1992) establishes nonparametric identification results under additional model restrictions (e.g., her assumptions W.2, W.4 and G.2).

different x_is. Regarding the choice of the starting point x_i^*, intuitively we should choose it in a way such that $\mathbb{C}_i^\infty$ is the largest. However, it can be shown that for any x_i' satisfying assumption N, if $x_i' \in \mathbb{C}_i^\infty$, then we will end up with the same $\mathbb{C}_i^\infty$; otherwise x_i' will lead to a non–overlapping set $\mathbb{C}_i^{\infty\prime}$.

The next corollary shows that the above iterative mechanism is not necessary if x_i^* provides the largest variations in player i's marginal choice probability conditional on X_i.

Assumption N. *(iii)* $\mathscr{S}_{\alpha_i(X)|X_i=x_i} \subseteq \mathscr{S}_{\alpha_i(X)|X_i=x_i^*}$ *for all* $x_i \in \mathscr{S}_{X_i}$.

Assumption N-(iii) requires X_{-i} to have sufficient variations conditional on $X_i = x_i^*$, which is satisfied in various situations. For instance, this is the case when $\mathscr{S}_{\alpha_i(X)|X_i=x_i^*}$ has full support $[0,1]$. See e.g. Wan and Xu (2014); Lewbel and Tang (2015).

Corollary 3. *Let* $S \in \mathcal{M}_2$. *Suppose assumptions SC, ER and N (i) to (iii) hold. Then the results in Proposition 6 hold, where* $\mathbb{C}_i^\infty = \{x_i \in \mathscr{S}_{X_i} : \text{ Rank of } \mathcal{R}_i(x_i) \geq 2^{I-1} - 2\}$.

There are normalizations other than assumption N. For instance, we can normalize two quantiles of the marginal distributions. Specifically, for $\tau_{i1}, \tau_{i2} \in \mathscr{S}_{\alpha_i(X)|X_i=x_i^*} \bigcap (0,1)$, we can set the quantiles $Q_{U_i}(\tau_{i1})$ and $Q_{U_i}(\tau_{i2})$ at some values, as long as (strict) monotonicity is satisfied. Proposition 6 still holds. Second, we can use the usual mean/variance normalization in binary variable models. This is possible under a full support condition. Namely, suppose $\mathcal{R}_i(x_i^*)$ has rank $2^{I-1} - 2$ and $(0,1) \subseteq \mathscr{S}_{\alpha_i(X)|X_i=x_i^*}$. Then we can set $\mathbb{E}(U_i) = 0$ and $\text{Var}(U_i) = 1$. Specifically, by Lemma 3, π_i is identified up to location and scale. Hence, all the quantiles Q_{U_i} are identified up to location and scale by (8). The latter are determined by the mean and variance normalization.

Lastly, we note that given the identification of the joint distribution of types, we might be interested in some additional structures on the error terms. For instance, suppose the private signals are affiliated in the sense of Milgrom and Weber (1982) as when $U_i = \xi + \epsilon_i$ where ξ is a common shock to all players and ϵ_i are iid (across players) idiosyncratic errors. Following Li and Vuong (1998), we can further deconvolute the joint distribution F_U to identify the marginal distributions of ξ and ϵ_i.

4.3. **Identification of $\mathcal{M}_3$.** Though not our focus of interest, $\mathcal{M}_3$ is nonparametrically identified in general. The argument does not essentially differ from that of $\mathcal{M}_2$: Assumption I only relaxes the support condition for identification of σ^*_{-i} in Lemma 2. We illustrate this in the next lemma.

Lemma 4. *Let $S \in \mathcal{M}_3$. Fix $x \in \mathscr{S}_X$. Then $\sigma^*_{-i}(\cdot|x, u^*_i(x))$ is identified by*

$$\sigma^*_{-i}(a_{-i}|x, u^*_i(x)) = \mathbb{P}\left(Y_{-i} = a_{-i}|X = x\right).$$

The proof is straightforward, hence omitted. By Proposition 2, we can also show that $\sigma^*_{-i}(a_{-i}|x, u^*_i(x)) = \mathbb{P}\left(Y_{-i} = a_{-i}|\alpha(X) = \alpha(x)\right)$. Similar results can be found in Pesendorfer and Schmidt-Dengler (2003); Aguirregabiria and Mira (2007); Bajari et al. (2010), among others. Further, the identification of π_i and Q_{U_i} in $\mathcal{M}_3$ follows Lemma 3 and Proposition 6 under assumptions ER and N. In particular, if $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ is not a singleton, then the rank condition requires that X_{-i} contain at least one continuous random variable to ensure there are sufficient variations in $\sigma^*_{-i}(\cdot|X, u^*_i(X))$ conditional on X_i and $\alpha_i(X)$ by varying X_{-i}. For a recent contribution using special regressors, see Lewbel and Tang (2015).

5. Discussion

In this section, we consider four issues related to our identification analysis. First, we illustrate how to nonparametrically estimate $\mathcal{M}_2$ based on the identification strategy established in Section 4.2. Second, without assumption ER, we examine the partial identification of $\mathcal{M}_2$. Third, we relax the single equilibrium assumption, i.e., we allow for multiple m.p.s. BNEs in the DGP, under which the observed data is a mixture of distributions from all these equilibria. Fourth, we discuss the issue of unobserved heterogeneity.

5.1. **A Sketch of Nonparametric Estimation.** To demonstrate how our nonparametric identification results can be used for estimation, we provide a sketch of a simple estimation procedure of a structure $\left[\pi; \{Q_{U_i}\}_{i=1}^I; C_U\right]$ in $\mathcal{M}_2$. A full development of nonparametric inference is beyond the scope of this paper.

For simplicity, we maintain the conditions in Corollary 3. Suppose the researcher observes an iid random sample $\{(X_1', Y_1')', \cdots, (X_n', Y_n')'\}$, where $X_t = (X_{1t}', \cdots, X_{It}')'$ and $Y_t =$

$(Y_{1t}, \cdots, Y_{It})'$ for $t = 1, \cdots, n$. Note that the number of players I is assumed to be constant for expositional simplicity. Here we suggest a flexible two–stage estimation procedure using sieve methods.

Step 1: Estimate the copula function C_U. For this step, we begin by estimating the marginal choice probability function $\alpha_i(\cdot)$. There are several nonparametric methods for such a purpose. Here we use sieve methods; see Chen (2007). Let F be a continuous distribution function with an interval support. For instance, we can choose $F = \Phi$, the standard Normal distribution. We then estimate $\alpha_i(\cdot)$ by $\hat{\alpha}_i(\cdot) = F(\hat{\gamma}_i(\cdot))$, where

$$\hat{\gamma}_i = \text{argsup}_{\gamma_i \in \Gamma_n} \frac{1}{n} \sum_{t=1}^{n} \left\{ Y_{it} \log F(\gamma_i(X_t)) + (1 - Y_{it}) \log \left[1 - F(\gamma_i(X_t))\right] \right\},$$

in which Γ_n is a Hölder class of real valued smooth basis functions mapping $\mathscr{S}_X$ to $\mathbb{R}$.

The estimation of C_U follows (5). Note that with C_U and $\alpha(X)$, we can obtain the conditional choice probability of $Y = a$ for any $a \in \mathcal{A}$ given X. Let $\mathbb{P}(Y = a|X) = G_I(a; C_U, \alpha(X))$ where G_I is a known function depending on I, C_U and $\alpha(X)$. For example, suppose $I = 2$. We can show that

$$\begin{aligned}
G_2((1,1); C_U, \alpha) &= C_U(\alpha), \\
G_2((0,1); C_U, \alpha) &= C_U(1, \alpha_2) - C_U(\alpha), \\
G_2((1,0); C_U, \alpha) &= C_U(\alpha_1, 1) - C_U(\alpha), \\
G_2((0,0); C_U, \alpha) &= 1 - C_U(\alpha_1, 1) - C_U(1, \alpha_2) + C_U(\alpha).
\end{aligned}$$

Further, let $\mathcal{C}_n$ be a Hölder class of "p–smooth" real valued basis functions mapping $[0,1]^I$ to $\mathbb{R}$ for some $p > 1$, which can approximate any square–integrable function arbitrarily well. Using the sieve MLE method, we then define our copula estimator by

$$\hat{C}_U = \text{argsup}_{C_U \in \mathcal{C}_n} \frac{1}{n} \sum_{t=1}^{n} \sum_{a \in \mathcal{A}} \mathbb{1}(Y_t = a) \log G_I(a; C_U, \hat{\alpha}(X)).$$

Consistency and asymptotic distribution of $\hat{\alpha}(\cdot)$ and $\hat{C}_U$ can be obtained from e.g. Chen (2007).[17] As the function estimator $\hat{C}_U$ might not be a proper copula, we can modify $\hat{C}_U$ by using a rearrangement approach similar to Chernozhukov et al. (2010b).

Step 2: Estimate the payoff functions π_i and quantile functions Q_{U_i}. Using Lemma 2, we first estimate the equilibrium beliefs σ^*_{-i} by

$$\hat{\sigma}^*_{-i}(a_{-i}|X_t, u_i^*(X_t)) = \frac{\partial G_I((1, a_{-i}); \hat{C}_U, \alpha)}{\partial \alpha_i}\Big|_{\alpha=\hat{\alpha}(X_t)}.$$

Next, we estimate π_i and Q_{U_i} from (8). Suppose $Q_{U_i} \in \mathcal{L}^2(0,1)$. Then, for a given complete orthonormal sequence $\{\psi_k, k \geq 1\}$ in $\mathcal{L}^2(0,1)$, we have

$$Q_{U_i}(\alpha) = \sum_{k=1}^{\infty} q_k^* \cdot \psi_k(\alpha), \;\; \forall \alpha \in \mathscr{S}_{\alpha_i(X)},$$

where $q_k^* = \int_0^1 \psi_k(s) \cdot Q_{U_i}(s) ds$. Let $\mathcal{Q}_n = \left\{ \sum_{k=1}^{K_n} q_k \psi_k : q_k \in \mathbb{Q}_k \right\}$ be a sieve space depending on the sample size, where $\mathbb{Q}_k \subset \mathbb{R}$ is compact. Note that (8) implies that π_i is identified up to Q_{U_i}, i.e.,

$$\pi_i(\cdot, x_i) = \left\{ \mathbb{E}\left[\Sigma^*_{-i}(X) \Sigma^{*\prime}_{-i}(X) \middle| X_i = x_i \right] \right\}^{-1} \mathbb{E}\left[\Sigma^*_{-i}(X) Q_{U_i}(\alpha_i(X)) \middle| X_i = x_i \right].$$

Hence, our estimator of Q_{U_i} is defined as follows:

$$\hat{Q}_{U_i} = \operatorname{arginf}_{Q_{U_i} \in \mathcal{Q}_n} \sum_{t=1}^{n} \Big[\sum_{a_{-i} \in \mathcal{A}_{-i}} \tilde{\pi}_i(a_{-i}, X_{it} | Q_{U_i}) \cdot \hat{\sigma}^*_{-i}(a_{-i}|X_t, u_i^*(X_t)) - Q_{U_i}(\hat{\alpha}_i(X_t)) \Big]^2$$

$$s.t. \;\; \tilde{\pi}_i\left(a^0_{-i}, x_i^* | Q_{U_i}\right) = 0, \;\; \text{and} \;\; \|\tilde{\pi}_i(\cdot, x_i^* | Q_{U_i})\| = 1,$$

[17] Alternatively, (5) can be used to propose a kernel regression estimator of the copula with generated regressors $\hat{\alpha}(X)$. See Mammen, Rothe and Schienle (2012).

where $\tilde{\pi}_i(\cdot,\cdot|Q_{U_i})$ is a functional of $Q_{U_i} = \sum_{k=1}^{K_n} q_k \psi_k$:

$$\begin{aligned}
&\tilde{\pi}_i(\cdot, X_{it}|Q_{U_i}) \\
&\quad \equiv \left[\sum_{s=1}^{n} \hat{\Sigma}^*_{-i}(X_s)\hat{\Sigma}^{*\prime}_{-i}(X_s)K\Big(\frac{X_{is}-X_{it}}{h}\Big)\right]^{-1}\left[\sum_{s=1}^{n} \hat{\Sigma}^*_{-i}(X_s)Q_{U_i}(\hat{\alpha}_i(X_s))K\Big(\frac{X_{is}-X_{it}}{h}\Big)\right] \\
&= \sum_{k=1}^{K_n} q_k \left\{\left[\sum_{s=1}^{n} \hat{\Sigma}^*_{-i}(X_s)\hat{\Sigma}^{*\prime}_{-i}(X_s)K\Big(\frac{X_{is}-X_{it}}{h}\Big)\right]^{-1}\left[\sum_{s=1}^{n} \hat{\Sigma}^*_{-i}(X_s)\psi_k(\hat{\alpha}_i(X_s))K\Big(\frac{X_{is}-X_{it}}{h}\Big)\right]\right\},
\end{aligned}$$

where K and h are a kernel function and bandwidth, respectively. Then, we let $\hat{\pi}_i(\cdot, x_i) = \tilde{\pi}_i(\cdot, x_i|\hat{Q}_{U_i})$. The proposed estimation procedure is easy to implement. Its precise asymptotic properties can be derived using the functional delta method in Van Der Vaart and Wellner (1996). As the quantile function estimator $\hat{Q}_{U_i}$ might not be strictly increasing, we can use Chernozhukov et al. (2010a)'s rearrangement approach to modify it, or choose a shape preserving sieve as suggested by e.g. Chen (2007).

5.2. **Partial Identification.** In this subsection, we study the partial identification of the game primitives when there are no exclusion restrictions, i.e. when assumption ER does not hold. It is worth emphasizing that the lack of point identification of a structure here is not due to multiple equilibria, but to the lack of identifying restrictions, i.e., the exclusion restrictions and the rank conditions. This is similar to, e.g., Shaikh and Vytlacil (2011) who study partial identification of the average structural function in a triangular model without imposing a restrictive support condition. When there are multiple m.p.s. BNEs, we still maintain the assumption of a single equilibrium being played for generating the distribution of observables.

By the same argument as for the identification of $\mathcal{M}_2$, the copula function C_U is point–identified on the extended support $\mathscr{S}^e_{\alpha(X)}$. Let $\mathscr{C}$ be the set of strictly increasing (on $(0,1]^I$) and continuously differentiable copula functions mapping $[0,1]^I$ to $[0,1]$. Then, the identification region of C_U can be characterized by

$$\mathscr{C}_I = \Big\{\widetilde{C}_U \in \mathscr{C} : \widetilde{C}_U(\alpha) = C_U(\alpha),\ \ \forall \alpha \in \mathscr{S}^e_{\alpha(X)}\Big\}.$$

For each $\widetilde{C}_U \in \mathscr{C}_I$, suppose we set $\widetilde{F}_{U_i}$ to be the uniform distribution on $[0,1]$ and $\tilde{\pi}_i(\cdot, x) = \alpha_i(x)$. Clearly, the constructed structure $[\tilde{\pi}; \widetilde{F}_U]$ is observationally equivalent to the underlying structure. Thus, $\mathscr{C}_I$ is the sharp identification region for C_U.

Next, we turn to the set identification of the quantile function Q_{U_i}. By assumption R, Q_{U_i} belongs to the set of strictly increasing and continuously differentiable functions mapping $[0,1]$ to $\mathbb{R}$, denoted as $\mathscr{Q}$. The next lemma shows that $\mathcal{M}_2$ imposes no restrictions on Q_{U_i} and its identification region is $\mathscr{Q}$.

Lemma 5. *Let $S \in \mathcal{M}_2$. For any $(\widetilde{Q}_{U_1}, \cdots, \widetilde{Q}_{U_I}) \in \mathscr{Q}^I$, there exists an observationally equivalent structure $\widetilde{S} \in \mathcal{M}_2$ with the marginal quantile function profile $(\widetilde{Q}_{U_1}, \cdots, \widetilde{Q}_{U_I})$.*

Now we discuss the sharp identification region for π_i. Let $\mathscr{G}$ be the set of functions mapping $\mathcal{A}_{-i} \times \mathscr{S}_X$ to $\mathbb{R}$.

Proposition 7. *Let $S \in \mathcal{M}_2$. Suppose assumption SC holds. Then the sharp identification region is given by $\left\{ [\widetilde{\pi}; \{\widetilde{Q}_{U_i}\}_{i=1}^I; \widetilde{C}_U] : (\widetilde{Q}_{U_i}, \widetilde{C}_U) \in (\mathscr{Q}, \mathscr{C}_I),\ \widetilde{\pi} \in \Theta_I(\{\widetilde{Q}_{U_i}\}_{i=1}^I, \widetilde{C}_U) \right\}$, where*

$$
\begin{aligned}
\Theta_I(\{\widetilde{Q}_{U_i}\}_{i=1}^I, \widetilde{C}_U) \equiv \Big\{ & \widetilde{\pi} \in \mathscr{G}^I : (a) \textit{ for all } x \in \mathscr{S}_X \textit{ and } i, \\
& \widetilde{Q}_{U_i}(\alpha_i(x)) = \sum_{a_{-i} \in \mathcal{A}_{-i}} \widetilde{\pi}_i(a_{-i}, x) \times \sigma^*_{-i}(a_{-i} | x, u_i^*(x)); (b) \textit{ for any m.p.s. profile } \delta : \\
& \mathbb{E}_\delta \left[\widetilde{\pi}_i (Y_{-i}, X) | X = x, U_i = \widetilde{Q}_{U_i}(\alpha_i) \right] - \widetilde{Q}_{U_i}(\alpha_i) \textit{ is weakly monotone in } \alpha_i \in (0,1) \Big\}.
\end{aligned}
$$

In the definition of Θ_I, condition (a) requires that $\pi_i(\cdot, x)$ should belong to a hyperplane, for which the slopes are given by the identified beliefs $\Sigma^*_{-i}(x)$; condition (b) does not impose much restriction on the structural parameters. Clearly, Θ_I is nonempty and convex.[18]

The identification region is unbounded and large. To see this, fix an arbitrary non–negative function $\kappa_i(x) \geq 0$. Let $\psi_i : \mathbb{R} \to \mathbb{R}$ satisfy: (i) ψ_i is a continuously differentiable and strictly increasing function; and (ii) for all x, $\kappa_i(x) Q_{U_i}(\alpha_i) - \psi_i(Q_{U_i}(\alpha_i))$ is weakly decreasing in $\alpha_i \in (0,1)$. Note that condition (ii) is equivalent to: $\inf_{u_i \in \mathscr{S}_{U_i}} \psi_i'(u_i) \geq \sup_{x \in \mathscr{S}_X} \kappa_i(x)$. Clearly, there are plenty of choices for such a function ψ_i. Let further $\widetilde{Q}_{U_i} = \psi_i(Q_{U_i})$ and $\tilde{\pi}_i(a_{-i}, x) = \xi_i(x) + \kappa_i(x) \times \pi_i(a_{-i}, x)$, in which $\xi_i(x) = \psi_i(Q_{U_i}(\alpha_i(x))) -$

[18] To see the nonemptiness, we can simply take $\widetilde{\pi}_i(\cdot, x) = \widetilde{Q}_{U_i}(\alpha_i(x))$.

$\kappa_i(x) \times Q_{U_i}(\alpha_i(x))$. Then, it can be verified that the constructed structure $[\widetilde{\pi}; \{\widetilde{Q}_{U_i}\}_{i=1}^I; \widetilde{C}_U]$ belongs to the identified set.[19] To narrow down the identification region, additional restrictions need to be introduced. Instead of imposing assumption ER, an alternative approach is to make assumptions on the payoff functional form. For instance, de Paula and Tang (2012) set $\pi_i(a_{-i}, x) = \pi_i^*(x) + g_i(a_{-i}) \times h_i^*(x)$, where g_i is a function known to all players as well as to the econometrician, and (π_i^*, h_i^*) are structural parameters in their model. Then, Proposition 7–(a) becomes: for all $x \in \mathscr{S}_X$ and i,

$$\widetilde{Q}_{U_i}(\alpha_i(x)) = \tilde{\pi}_i^*(x) + \tilde{h}_i^*(x) \times \sum_{a_{-i} \in \mathcal{A}_{-i}} g_i(a_{-i}) \times \sigma_{-i}^*(a_{-i}|x, u_i^*(x)),$$

which imposes a linear restriction on $\tilde{\pi}_i^*(x)$ and $\tilde{h}_i^*(x)$ by noting that $\sum_{a_{-i} \in \mathcal{A}_{-i}} g_i(a_{-i}) \times \sigma_{-i}^*(a_{-i}|x, u_i^*(x))$ is identified under the conditions in Lemma 2. Moreover, Proposition 7–(b) imposes an additional restriction on the copula function $\widetilde{C}_U$.

When $\mathscr{S}_{\alpha(X)} = (0,1)^I$, $\mathscr{C}_I$ degenerates to the singleton $\{C_U\}$. In this case, the sharp identification region for $(\pi, \{Q_{U_i}\}_{i=1}^I)$ can be characterized in a more straightforward manner:

$$\begin{aligned}\Theta_I^* = \Big\{ &(\widetilde{\pi}, \{\widetilde{Q}_{U_i}\}_{i=1}^I) \in \mathscr{G}^I \times \mathscr{Q}^I : (a') \text{ for all } x \in \mathscr{S}_X \text{ and } i, \\ &\widetilde{Q}_{U_i}(\alpha_i(x)) = \sum_{a_{-i} \in \mathcal{A}_{-i}} \widetilde{\pi}_i(a_{-i}, x) \times \sigma_{-i}^*(a_{-i}|x, u_i^*(x)); (b') \text{ and for all } \alpha_{-i} \in [0,1]^{I-1}, \\ &\sum_{a_{-i} \in \mathcal{A}_{-i}} \widetilde{\pi}_i(a_{-i}, x) \times \sigma_{-i}^{\alpha}(a_{-i}, \alpha_{-i}, \alpha_i) - \widetilde{Q}_{U_i}(\alpha_i) \text{ is weakly monotone in } \alpha_i \in (0,1) \Big\},\end{aligned}$$

where $\sigma_{-i}^{\alpha}(a_{-i}, \alpha_{-i}, \alpha_i) = \mathbb{P}_{C_U}(C_j \leq \alpha_j \ \forall a_j = 1; C_j > \alpha_j \ \forall a_j = 0 | C_i = \alpha_i)$.

5.3. **Multiple Equilibria in DGP.** The problems raised by multiple equilibria have a long history in economics. See e.g. Jovanovic (1989) for empirical implications of multiple equilibria, and Morris and Shin (2001) for a recent discussion in macroeconomic modeling.

[19] Note that the payoff normalization imposed in Proposition 6 is not helpful to bound the payoffs, since it applies only at one point $x_i^* \in \mathscr{S}_{X_i}$. Specifically, if one imposes a similar normalization on $\pi_i(\cdot, x^*)$ for some $x^* \in \mathscr{S}_X$, we would need to restrict the monotone mapping ψ_i to satisfy $\psi_i(\alpha_i(x^*)) = \alpha_i^*(x)$ when constructing an observational equivalence structure. Nevertheless, without assumption ER, the unboundedness of the partially identified set still holds for all $x \in S_X$ satisfying $\alpha_i(x) \neq \alpha_i(x^*)$.

The static games literature have struggled with difficulties arising from equilibrium multiplicity since the mid 1980s. See e.g. Bjorn and Vuong (1984). Researchers have developed essentially three approaches. In the first approach, one assumes there is a single equilibrium in the DGP as we do. See e.g. Aguirregabiria and Nevo (2013) for a survey. Sometimes this assumption is satisfied when the model admits a unique equilibrium. Pesendorfer and Schmidt-Dengler (2003) provide empirical justifications for this assumption, in particular, when data come from the same game repeatedly played across different time periods. See also e.g. Bajari et al. (2010). A more sophisticated solution is to identify a subset in the support of covariates that admit a unique equilibrium. See Xu (2014) in a parametric setting.

In the second approach, a seminal paper by Tamer (2003) introduces partial identification analysis in a discrete game of complete information. This allows us to bound the parameters of interest without specifying which equilibrium is chosen. In a parametric model, Aradillas-Lopez and Tamer (2008) obtain inequality constrains by exploiting level–k rationality in either a complete or incomplete information framework. In a semiparametric setting with incomplete information, Wan and Xu (2014) develop upper/lower bounds for equilibrium beliefs to achieve point identification of payoff parameters under a full support condition on regressors. In the third approach, one introduce a probability distribution λ over the set of equilibria. See e.g. Bjorn and Vuong (1984); Bajari et al. (2010) for complete information games, and Aguirregabiria and Mira (2007) for incomplete information games in parametric settings.

In general, the issue of multiple equilibria is a largely unexplored area of research in a nonparametric framework, which is considered in this paper. We first address how to detect multiple equilibria in our setting. Next we discuss the problem of identification in the presence of multiple equilibria. Our discussion below focuses on $\mathcal{M}_2$, since $\mathcal{M}_1$ imposes almost no restrictions by Proposition 1.[20]

In $\mathcal{M}_2$, we can detect multiple equilibria from the model restrictions derived in Proposition 2, specifically, restrictions R1 and R2. This is because, in the presence of multiple equilibria in the data, R1 and/or R2 are violated in general. Moreover, if assumption ER

[20] In $\mathcal{M}_3$ or $\mathcal{M}_3'$, multiple equilibria can be detected by testing the conditional independence as shown in Proposition 3 and corollary 2. See also de Paula and Tang (2012).

holds, (8) introduces additional model restrictions, providing stronger power to detect the existence of multiple equilibria. To see this, we first fix $X_i = x_i$. Under assumption ER, (8) can be rewritten as

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \times \sigma^*_{-i}(a_{-i}|x, u^*_i(x)) - Q_{U_i}(\alpha_i(x)) = 0, \tag{10}$$

for all $x \in \mathscr{S}_{X|X_i=x_i}$. Suppose $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ is not a singleton. Then, there are strategic effects. In addition, conditioning on $\alpha_i(X) = \alpha_i \in \mathscr{S}_{\alpha_i(X)|X_i=x_i}$, the random vector $\Sigma^*_{-i}(X)$, which is a probability mass function on $\mathcal{A}_{-i}$, has to be distributed on a hyperplane in $\mathbb{R}^{2^{I-1}}$. In addition, the slope $\pi_i(\cdot, x_i)$ of the hyperplane remains constant as α_i varies, while its intercept $Q_{U_i}(\alpha_i(x))$ strictly increases in α_i. Because the equilibrium beliefs $\sigma^*_{-i}(\cdot|x, u^*_i(x))$ are identified under assumption SC, violations of these restrictions indicate the presence of multiple equilibria. In the special case of $I = 2$, these restrictions imply that, conditional on $X_i = x_i$, $\alpha_j(x)$ is a monotone function of $\alpha_i(x)$. Violations of such a monotonicity indicates multiple equilibria.

Next, to relax the single equilibrium assumption for identification analysis, we follow Henry et al. (2014) by introducing an instrumental variable Z, which does not affect players' payoffs, the distribution of types, or the set of equilibria in the game, but can effectively change the equilibrium selection. For each $x \in \mathscr{S}_X$, let $\mathscr{E}(x)$ be the set of m.p.s. BNEs in the game with $X = x$. Note that $\mathscr{E}(x)$ could be an infinite collection and the number of equilibria depends on the value of x. To simplify, we assume that players only focus on a subset $\Gamma(x)$ of $\mathscr{E}(x)$ for the DGP, i.e., the set of equilibria that will be played in the data. We further assume that the number of elements in $\Gamma(x)$ is finite and bounded above by a constant J ($J \geq 2$) for all x.

Let λ be a probability distribution $\{p^\lambda_1, \cdots, p^\lambda_J\}$ on the support $\{1, \cdots, J\}$ such that the j–th equilibrium occurs with probability p^λ_j. The distribution λ may have some zero mass points, which means that the number of equilibria in $\Gamma(x)$ is strictly less than J. Essentially, λ summarizes the mixture of equilibrium distributions arising from the equilibrium selection mechanism. Following Henry et al. (2014), we assume that the probability distribution λ varies with X and Z, where Z is a vector of instrumental variables that does not affect either $\mathscr{E}(X)$ or $\Gamma(X)$, but has influence on the equilibrium selection through λ.

In Henry et al. (2014), it is shown that the set of component distributions is partially identified in the space of probability distributions. For example, for $J = 2$, the observed distribution $F_{Y|X=x}$ is a convex combination of the two component distributions generated from the two equilibria in $\Gamma(x)$. Then, variations of the instrumental variable Z cause the mixture distribution to move along a straight line in the function space of probability distributions. Further, we can point identify the set of distributions corresponding to $\Gamma(x)$ if Z has sufficient variations. To see this, w.l.o.g. let $J = 2$. For each $x \in \mathscr{S}_X$, suppose there exist some (unknown) $z, z' \in \mathscr{S}_{Z|X=x}$ such that $\lambda = (0, 1)$ when $(X, Z) = (x, z)$ and $\lambda = (1, 0)$ when $(X, Z) = (x, z')$. In the space of probability distributions, the two equilibrium distributions can be identified as the two extreme points of the convex hull (which is a straight line) of the collection of distributions $F_{Y|X=x,Z=z}$ for all $z \in \mathscr{S}_{Z|X=x}$. Either one of them represents a probability distribution from a single m.p.s. BNE given x, which thereafter provides the identification of the underlying game structure as we discussed in the identification section.

5.4. **Correlated Types vs Unobserved Heterogeneity with Independent Types.** Within a paradigm where private signals are independent unconditionally or conditionally given X, a known approach for generating correlation among actions given X is to introduce unobserved heterogeneity. In a fully parametric setting, Aguirregabiria and Mira (2007) and Grieco (2014) introduce unobserved heterogeneity through some payoff relevant variables ζ publicly observed by all players, but not by the researcher. An important question is whether one can distinguish this model from our model with correlated types. Because $\mathcal{M}_1$ can rationalize any distributions generated by a model with unobserved heterogeneity and independent types, we consider $\mathcal{M}_2$ below.

Consider the following payoffs with unobserved heterogeneity:

$$\Pi_i(Y, X, \zeta, U_i) = \begin{cases} \pi_i(Y_{-i}, X, \zeta) - U_i, & \text{if } Y_i = 1, \\ 0, & \text{if } Y_i = 0, \end{cases}$$

where ζ is a discrete variable that is unobserved to the researcher. Let the support of ζ be $\{z_1, \cdots, z_J\}$ and $p_j(x) = \mathbb{P}(\zeta = z_j | X = x)$. To simplify, we assume there are only two players with $U_1 \perp U_2$ and $(U_1, U_2) \perp (X, \zeta)$. We assume further that there is only a single equilibrium in the DGP for every $(x, z_j) \in \mathscr{S}_{X\zeta}$, which is also assumed in our model $\mathcal{M}_2$. Then, the joint choice probability for the model with unobserved heterogeneity and independent types is

$$\mathbb{E}(Y_1Y_2|X = x) = \sum_{j=1}^{J} \mathbb{E}(Y_1Y_2|X = x, \zeta = z_j) \cdot p_j(x) = \sum_{j=1}^{J} \alpha_1(x, z_j) \cdot \alpha_2(x, z_j) \cdot p_j(x)$$

where $\alpha_i(x, z_j) = \mathbb{E}(Y_i|X = x, \zeta = z_j)$. Moreover, the marginal choice probability is

$$\alpha_i(x) = \sum_{j=1}^{J} \alpha_i(x, z_j) \cdot p_j(x), \quad \text{for } i = 1, 2,$$

where $\alpha_i(x) = \mathbb{E}(Y_i|X = x)$.

We now argue that these joint and marginal choice probabilities can violate restriction R1 in Proposition 2. Specifically, R1 requires that $\mathbb{E}(Y_1Y_2|X = x)$ be a function of $(\alpha_1(x), \alpha_2(x))$ only. The model with unobserved heterogeneity and independent types does not exclude the possibility that there exist $x, x' \in \mathscr{S}_X$ such that $\alpha_i(x) = \alpha_i(x')$ for $i = 1, 2$, but $\mathbb{E}(Y_1Y_2|X = x) \neq \mathbb{E}(Y_1Y_2|X = x')$. For instance, suppose $J = 2$, $p_j(x) = p_j(x') = 0.5$ for $j = 1, 2$, $\alpha_1(x, z_1) = \alpha_2(x, z_1) = 0.4$, $\alpha_1(x, z_2) = \alpha_2(x, z_2) = 0.6$, $\alpha_1(x', z_1) = 0.3$, $\alpha_2(x', z_1) = 0.7$, $\alpha_1(x', z_2) = 0.7$, $\alpha_2(x', z_2) = 0.3$. Therefore, $\alpha_1(x) = \alpha_1(x') = 0.5$ and $\alpha_2(x) = \alpha_2(x') = 0.5$, but $\mathbb{E}(Y_1Y_2|X = x) = 0.26 \neq \mathbb{E}(Y_1Y_2|X = x') = 0.21$. Consequently, the model with unobserved heterogeneity and independent types can be distinguished from model $\mathcal{M}_2$. We have focused on R1 above, but monotonicity in $\alpha_i(X)$ (see R2) could be violated as well for similar reasons.

6. Conclusion

This paper studies the rationalization and identification of discrete games with correlated types within a fully nonparametric framework. Allowing for correlation across types is important in global games and in models with social interactions as it represents correlated information and homophily, respectively. Regarding rationalization, we show that our baseline game–theoretical model $\mathcal{M}_1$ with a single m.p.s. BNE in the DGP does not impose any essential restrictions on observables, and hence is not testable in view of players' choice probabilities only. We also show that exogeneity is testable, because R1–R3 in Proposition 2 characterize all the restrictions imposed by exogeneity. For instance, we can view R1 as a regression of the joint choices on covariates that depends on the latter only through the marginal choice probabilities. Thus, to test R1 we can extend Fan and Li (1996) and Lavergne and Vuong (2000) significance tests by allowing for estimation of the marginal choice probabilities. Moreover, R2 is a monotonicity restriction that can be tested by testing the convexity of its integral, see e.g., Delgado and Escanciano (2012).[21]

Model $\mathcal{M}_3$ is mostly adopted in empirical work within a parametric or semiparametric setting. We show that all its restrictions reduce to the mutual independence of choices conditional on covariates. This can be tested by using conditional independence tests, see e.g. Su and White (2007, 2008). Moreover, Proposition 3 and Corollary 2 show that the same restriction characterizes model $\mathcal{M}'_3$ which only assumes mutual independence of types conditional on covariates and a single (not necessary monotone) BNE in the DGP. These two assumptions seem to be unrelated, but actually are two sides of the same coin. Maintaining a single equilibrium in the DGP, we can use the mutual independence of choices given covariates to test mutual independence of types which is widely assumed in the literature. On the other hand, maintaining mutual independence of types, we can use the same mutual independence of choices to test for a single equilibrium versus multiple equilibria. See, e.g., de Paula and Tang (2012).

It is worth noting that the above tests do not rely on identification and consequently on the assumptions used to identify the primitives of the various models. In particular, we show that model $\mathcal{M}_2$ is identified up to a single location–scale normalization under exclusion restrictions, rank conditions and a non–degenerate support condition. The exclusion restrictions take the form of excluding part of a player's payoff shifters from all other players' payoffs as frequently assumed in the literature. Specifically, the dependence of players' joint choices on the marginal choice probabilities identifies the dependence across types, while the dependence of a player's marginal choice probabilities on her equilibrium beliefs identifies her payoffs. Without exclusion restrictions, we show that the sharp identification region of players' payoffs is unbounded.

[21] In the context of finite normal form games, the Quantal Response Equilibrium has an identical structure to BNE in our setting. In a semiparametric setting, Melo et al. (2014) obtain model restrictions characterized by monotonicity, which are similar to R1, and then propose a moment inequality test.

Our identification results are useful for estimation of global games and social interaction models. In a semiparametric setup, Liu and Xu (2012) propose an estimation procedure for our model $\mathcal{M}_2$ with linear payoffs, and establish the root–n consistency of the linear payoff coefficients. A fully nonparametric estimation deserves to be studied following the estimation sketch given in Section 5.1. Specifically, we could rely on the identification results and propose sample–analog estimators for the players' payoffs and the joint distribution of private information. The equilibrium condition (8) is the key estimating equation. It has the nice feature to be partially linear, namely, linear in the payoffs and nonparametric in the quantile. A difficulty is to take into account the estimation of the beliefs of the player at the margin and the marginal choice probabilities. Part of the problem could be addressed by using the recent literature on nonparametric regression with generated covariates, see e.g. Mammen et al. (2012). An important question is to determine the optimal (best) rate at which the primitives of $\mathcal{M}_2$ can be estimated from players' choices. Proposition 2 which characterizes all the restrictions imposed by $\mathcal{M}_2$ will be useful, see e.g. Guerre et al. (2000).

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Appendix A. Existence of m.p.s. BNEs

A.1. **Proof of Lemma 1.** First, we show the existence of m.p.s. BNE. Assumptions G1–G6 of Reny (2011) are satisfied in our discrete game under assumption R. Moreover, by assumption M, when other players employ m.p.s., player i's best response is also a joint–closed set of m.p.s.. By Reny (2011, Theorem 4.1), the conclusion follows.

We now show the second half. Fix $X = x$. Because $\sigma^*_{-i}(a_{-i}|x, u_i)$ are continuous in u_i under assumption R, then $\sum_{a_{-i}} \pi_i(a_{-i}, x)\sigma^*_{-i}(a_{-i}|x, u_i) - u_i$ is a continuously decreasing function in u_i.

Suppose $\underline{u}_i(x) < u^*_i(x) < \overline{u}_i(x)$. It follows that

$$\sum\nolimits_{a_{-i}} \pi_i(a_{-i}, x)\sigma^*_{-i}(a_{-i}|x, u^*_i(x)) - u^*_i(x) = 0.$$

Hence, conditional on $\underline{u}_i(X) < u^*_i(X) < \overline{u}_i(X)$, we have

$$Y_i = \mathbf{1}\left[U_i \le u^*_i(X)\right] = \mathbf{1}\left[U_i \le \sum_{a_{-i}} \pi_i(a_{-i}, X)\sigma^*_{-i}\big(a_{-i}|X, u^*_i(X)\big)\right].$$

Suppose $u^*_i(x) = \overline{u}_i(x)$. Then $\sum_{a_{-i}} \pi_i(a_{-i}, x)\sigma^*_{-i}(a_{-i}|x, \overline{u}_i(x)) - \overline{u}_i(x) \ge 0$, which implies that conditional on $u^*_i(X) = \overline{u}_i(X)$, there is

$$Y_i = \mathbf{1}\left[U_i \le \overline{u}_i(X)\right] \le \mathbf{1}\left[U_i \le \sum_{a_{-i}} \pi_i(a_{-i}, X)\sigma^*_{-i}\big(a_{-i}|X, \overline{u}_i(X)\big)\right].$$

Because $\mathbf{1}\left[U_i \le \overline{u}_i(X)\right] = 1$ a.s., thus

$$Y_i = \mathbf{1}\left[U_i \le \overline{u}_i(X)\right] = \mathbf{1}\left[U_i \le \sum_{a_{-i}} \pi_i(a_{-i}, X)\sigma^*_{-i}\big(a_{-i}|X, \overline{u}_i(X)\big)\right] \quad a.s.$$

Similar arguments hold for the case $u^*_i(X) = \underline{u}_i(X)$.

□

A.2. **Existence of m.p.s. BNEs under primitive conditions.**

Definition 3. *a set $A \subseteq \mathbb{R}^d$ is upper if and only if its indicator function is non–decreasing, i.e., for any $x, y \in \mathbb{R}^d$, $x \in A$ and $x \le y$ imply $y \in A$, where $x \le y$ means $x_i \le y_i$ for $i = 1, \cdots, d$.*

Assumption PRD (Positive Regression Dependence)**.** *For any $x \in \mathscr{S}_X$ and any upper set $A \subseteq \mathbb{R}^{I-1}$, the conditional probability $\mathbb{P}\left(U_{-i} \in A | X = x, U_i = u_i\right)$ is non–decreasing in $u_i \in \mathscr{S}_{U_i|X=x}$.*

Assumption SCP (Strategic Complement Payoffs)**.** *For any $x \in \mathscr{S}_X$ and $u_i \in \mathscr{S}_{U_i|X=x}$, suppose $a_{-i} \le a'_{-i}$, then $\pi_i(a_{-i}, x) \le \pi_i(a'_{-i}, x)$.*

Lemma 6. *Suppose assumptions R, PRD and SCP hold. For any $x \in \mathscr{S}_X$, there exists an m.p.s. BNE.*

Proof. By Lemma 1, it suffices to show that assumption M holds. Fix $x \in \mathscr{S}_X$. Given an arbitrary m.p.s. profile: for $i = 1, \cdots, I$, $\delta_i(x, u_i) = \mathbf{1}[u_i \leq u_i(x)]$, where $u_i(\cdot)$ is arbitrarily given. By assumptions PR and SCP, and Lehmann (1955), for any $u_i < u_i'$ in the support, we have

$$\mathbb{E}_\delta \left[\pi_i(Y_{-i}, X)|X = x, U_i = u_i'\right] \leq \mathbb{E}_\delta \left[\pi_i(Y_{-i}, X)|X = x, U_i = u_i\right].$$

Thus, $\mathbb{E}_\delta \left[\pi_i(Y_{-i}, X)|X = x, U_i = u_i\right] - u_i$ is a weakly decreasing function of u_i. □

Appendix B. Rationalization

B.1. **Proof of Proposition 1.** Prove the "only if part" first: Proofs by contradiction. Let $F_{Y|X}$ be rationalized by $\mathcal{M}_1$, i.e., some $S \in \mathcal{M}_1$ can generate $F_{Y|X}$. Fix $X = x$ and let equilibrium be characterized by $(u_1^*(x), \cdots, u_I^*(x))$. For some $a \in \mathcal{A}$, w.l.o.g., $a = (1, \cdots, 1)$, suppose $\mathbb{P}(Y = a|X = x) = 0$ and $\mathbb{P}(Y_i = a_i|X = x) > 0$ for all i. It follows that $\mathbb{P}(U_1 \leq u_1^*(x), \cdots, U_I \leq u_I^*(x)|X = x) = 0$ and $\mathbb{P}(U_i \leq u_i^*(x)|X = x) > 0$ for all i, which violates assumption R. Then $S \notin \mathcal{M}_1$. Contradiction.

Proofs for the "if part": Fix an arbitrary $x \in \mathscr{S}_X$. First, we assume $\mathbb{P}(Y = a|X = x) > 0$ for all $a \in \mathcal{A}$, which will be relaxed later. Now we construct a structure in $\mathcal{M}_1$ that will lead to $F_{Y|X}(\cdot|x)$.

Let $\pi_i(a_{-i}, x) = \alpha_i(x)$ for $i = 1, \cdots, I$. Note that there is no strategic effect by construction and assumption M is satisfied. Now we construct $F_{U|X}(\cdot|x)$. Let $F_{U_i|X}(\cdot|x)$ be uniformly distributed on $[0, 1]$. So it suffices to construct the copula function $C_{U|X}(\cdot|x)$ on $[0, 1]^I$. We first construct $C_{U|X}(\cdot|x)$ on a finite sub–support: $\{\mathbb{E}(Y_1|X = x), 1\} \times \cdots \times \{\mathbb{E}(Y_I|X = x), 1\}$. Then we extend it to a proper copula function with the full support $[0, 1]^I$. Let $C_{U|X}(\alpha_1, \cdots, \alpha_I|x) = \mathbb{E}(\prod_{j=1}^p Y_{i_j}|X = x)$ where $i_1, \cdots, i_p$ are all the indexes such that $\alpha_{i_j} = \mathbb{E}(Y_{i_j}|X = x)$; while other indexes have $\alpha_k = 1$. Because $\mathbb{P}(Y = a|X = x) > 0$ for all $a \in \mathcal{A}$, $C_{U|X}(\cdot|x)$ is strictly increasing in each index on the finite sub–support. Thus it is straightforward that we can extend $C_{U|X}(\cdot|x)$ to the whole support $[0, 1]^I$ as a strictly increasing (on the support $(0, 1]^I$) and smooth copula function. By construction, it is straightforward that the constructed structure can generate $F_{Y|X}(\cdot|x)$.

When $\mathbb{P}(Y = a|X = x) = 0$ for some a's in $\mathcal{A}$. By the condition in Proposition 1, the conditional distribution of Y given $X = x$ is degenerated in some indexes. W.l.o.g., let $\{1, \cdots, k\}$ be set of indexes such that $\mathbb{P}(Y_i = 1|X = x) = 0$ or 1, and let $\{k + 1, \cdots, I\}$ satisfy $0 < \mathbb{P}(Y_i = 1|X = x) < 1$. Then let again $\pi_i(a_{-i}, x) = \alpha_i(x)$ for $i = 1, \cdots, I$. For player $i = k + 1, \cdots, I$, we can construct a sub–copula function $C_{U_{k+1}, \cdots, U_I|X}(\cdot|x)$ as described above such that $C_{U_{k+1}, \cdots, U_I|X}(\cdot|x)$ is strictly increasing and smooth. Further, we can extend $C_{U_{k+1},\cdots,U_I|X}(\cdot|x)$ to a proper copula function having the full support $[0,1]^I$. Similarly, the constructed structure generates $F_{Y|X}(\cdot|x)$. $\square$

B.2. **Rationalizing All Probability Distributions.** Suppose we replace assumption R with the following conditions in Reny (2011): For every $x \in \mathscr{S}_X$,

G.2. The distribution $F_{U_i|X}(\cdot|x)$ on $\mathscr{S}_{U_i|X=x}$ is atomless.

G.3. There is a countable subset $\mathscr{S}^0_{U_i|X=x}$ of $\mathscr{S}_{U_i|X=x}$ such that every set in $\mathscr{S}_{U_i|X=x}$ assigned positive probability by $F_{U_i|X}(\cdot|x)$ contains two points between which lies a point in $\mathscr{S}^0_{U_i|X=x}$.

Note that it is straightforward that assumptions G.1 and G.4 through G.6 in Reny (2011) are all satisfied in our discrete game because the action space $\mathcal{A}$ is finite and the conditional distribution of U given $X = x$ has a hypercube support in $\mathbb{R}^I$. Thus, the conclusion in Lemma 1 still holds (i.e., existence of an m.p.s. BNE) under assumptions G.2, G.3 and M. Moreover, let $\mathcal{M}'_1 \equiv \{S : \text{G.2, G.3 and M hold and a single m.p.s. BNE is played}\}$. Then, we generalize Proposition 1.

Lemma 7. *Any conditional distribution $F_{Y|X}$ can be rationalized by $\mathcal{M}'_1$.*

Proof. We prove by construction. Fix x. Let $\pi_i(a_{-i}, x) = \alpha_i(x)$ for all i. Note that there is no strategic effect by construction and assumption M is satisfied. Now we construct $F_{U|X}(\cdot|x)$. Let $[0,1]^I$ be the support of the distribution and partition it into 2^I disjoint events: $\bigotimes_{i=1}^I\{[0,\alpha_i(x)),[\alpha_i(x),1]\}$ [22]. Further, we define a conditional distribution $F_{U|X=x,U\in B_j}$ as a uniform distribution on B_j, where B_j is the j–th event in the partition of the support. Moreover, let $\mathbb{P}(U \in B_j|X = x) = \mathbb{P}(Y = a(j)|X = x)$ where $a(j) \in \mathcal{A}$ and satisfies $a_i(j) = 0$ if the i–th argument of event B_j is $[\alpha_i(x), 1]$, and $a_i(j) = 1$ if the i–th argument is $[0, \alpha_i(x))$. With such construction, the marginal distribution of U_i given $X = x$ is a uniform distribution on $[0, 1]$ which satisfies assumptions G.2 and G.3. It can be verified that the constructed structure leads to $F_{Y|X}(\cdot|x)$. $\square$

B.3. **Proof of Proposition 2.**

[22] To have meaningful partition, it is understood that $\{[0,\alpha_i(x)),[\alpha_i(x),1]\}$ becomes $\{\{0\},(0,1]\}$ when $\alpha_i(x) = 0$.

Proof. We first show the "only if part". Suppose that the distribution $F_{Y|X}(\cdot|\cdot)$ rationalized by $\mathcal{M}_1$ is derived from $\widetilde{S} = [\tilde{\pi}; \tilde{F}_{U|X}] \in \mathcal{M}_2$. Then

$$\mathbb{E}\big(\prod_{j=1}^{p} Y_{i_j}|X\big) = \mathbb{P}\big(Y_{i_1} = 1, \cdots, Y_{i_p} = 1|X\big)$$
$$= \mathbb{P}\big(U_{i_1} \le \tilde{u}^*_{i_1}(X), \cdots, U_{i_p} \le \tilde{u}^*_{i_p}(X)|X\big) = \widetilde{C}_{U_{i_1},\cdots,U_{i_p}}\big(\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)\big).$$

Similarly,

$$\mathbb{E}\big(\prod_{j=1}^{p} Y_{i_j}|\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)\big) = \widetilde{C}_{U_{i_1},\cdots,U_{i_p}}\big(\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)\big).$$

Thus, we have condition R1. Further, R2 and R3 obtain by the properties of the copula function $\widetilde{C}_{U_{i_1},\cdots,U_{i_p}}$.

Proofs for the "if part". For any $x \in \mathscr{S}_X$, let $\tilde{\pi}_i(\cdot, x) = \alpha_i(x)$. Let $\tilde{F}_{U_i}$ denote the CDF of uniform distribution on $[0, 1]$. For all $1 \le i_1 < \cdots < i_p \le I$, $(\alpha_{i_1}, \cdots, \alpha_{i_p}) \in \mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}$ and $x \in \mathscr{S}_X$, define $\widetilde{F}_{U_{i_1},\cdots,U_{i_p}}(\cdot, \cdots, \cdot)$ as follows: for each $\alpha_{i_1}, \cdots, \alpha_{i_p} \in \mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}$,

$$\widetilde{F}_{U_{i_1},\cdots,U_{i_p}}(\alpha_{i_1}, \cdots, \alpha_{i_p}) = \mathbb{E}\Big[\prod_{j=1}^{p} Y_{i_j}\big|\alpha_{i_1}(X) = \alpha_{i_1}, \cdots, \alpha_{i_p}(X) = \alpha_{i_p}\Big].$$

Thus, we define $\widetilde{F}_U$ on the support $\{\alpha : (\alpha_{i_1}, \cdots, \alpha_{i_p}) \in \mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}; \text{other } \alpha_{i_j} = 1\}$.

By Proposition 1, we have that $\widetilde{F}_{U_{i_1},\cdots,U_{i_p},U_k}(\alpha_{i_1}, \cdots, \alpha_{i_p}, \alpha_k) < \widetilde{F}_{U_{i_1},\cdots,U_{i_p}}(\alpha_{i_1}, \cdots, \alpha_{i_p})$ for any $k \ne i_j$, $j = 1, \cdots, p$, $\alpha_{i_j} > 0$ and $\alpha_k < 1$. Further, under conditions R2, R3, $\widetilde{F}_U$ is strictly increasing and continuously differentiable on $\{\alpha : (\alpha_{i_1}, \cdots, \alpha_{i_p}) \in \mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}; \text{other } \alpha_{i_j} = 1\}$. Hence, we can extend it to the whole support $[0, 1]^I$ as a proper distribution function such that it is strictly increasing and continuously differentiable on $[0, 1]^I$. The extended $\widetilde{F}_U(\cdot)$ will yield a positive and continuous conditional Radon–Nikodym density on $[0, 1]^I$.

By construction, $[\tilde{\pi}; \widetilde{F}_U] \in \mathcal{M}_2$. Fix $X = x$. The constructed structure $[\tilde{\pi}; \tilde{F}_U(\cdot)]$ will generate the given marginal distribution $\alpha_i(x)$ for all i. Moreover, for any tuple $\{i_1, \cdots, i_p\}$ from $\{1, \cdots, I\}$,

$$\widetilde{\mathbb{P}}(Y_{i_1} = 1, \cdots, Y_{i_p} = 1|X = x) = \widetilde{F}_{U_{i_1},\cdots,U_{i_p}}\Big(\alpha_{i_1}(x), \cdots, \alpha_{i_p}(x)\Big)$$
$$= \mathbb{E}\Big[\prod_{j=1}^{p} Y_{i_j}\big|\alpha_{i_1}(X) = \alpha_{i_1}(x), \cdots, \alpha_{i_p}(X) = \alpha_{i_p}(x)\Big] = \mathbb{E}\Big[\prod_{j=1}^{p} Y_{i_j}\big|X = x\Big].$$

Because the tuple $\{i_1, \cdots, i_p\}$ is arbitrary, then $[\tilde{\pi}, \widetilde{F}_U]$ generates the distribution $F_{Y|X}(\cdot|x)$. □

B.4. **Proof of Proposition 3.**

Proof. The "only if part" follows directly from assumption I and the single equilibrium condition. It suffices to show the "if part".

Fix a distribution $F_{Y|X}$ that satisfies the condition. Let $\tilde{F}_{U_i|X} = \tilde{F}_{U_i}$ be a uniform distribution on $[0,1]$ and $\tilde{F}_{U|X} = \prod_{i=1}^{I} \tilde{F}_{U_i}$. Moreover, let $\tilde{\pi}_i(\cdot, x) = \alpha_i(x)$ for any $x \in \mathscr{S}_X$. By construction, $[\tilde{\pi}; \tilde{F}_{U|X}]$ satisfies assumptions R, M, E, and I. Hence, $[\tilde{\pi}; \tilde{F}_{U|X}] \in \mathcal{M}_3$.

It suffices to show that the constructed structure $[\tilde{\pi}; \tilde{F}_{U|X}]$ can generate $F_{Y|X}$. Fix x. By construction, we have that $\widetilde{\mathbb{P}}(Y_i = 1|X = x) = \alpha_i(x)$. Moreover, for any tuple $\{i_1, \cdots, i_p\}$ from $\{1, \cdots, I\}$,

$$\widetilde{\mathbb{P}}(Y_{i_1} = 1, \cdots, Y_{i_p} = 1|X = x) = \widetilde{F}_{U_{i_1}, \cdots, U_{i_p}}\left(\alpha_{i_1}(x), \cdots, \alpha_{i_p}(x)\right)$$
$$= \prod_{j=1}^{p} \alpha_{i_j}(x) = \mathbb{P}(Y_{i_1} = 1, \cdots, Y_{i_p} = 1|X = x).$$

Because the tuple $\{i_1, \cdots, i_p\}$ is arbitrary, then $[\tilde{\pi}, \widetilde{F}_{U|X}]$ generates the distribution $F_{Y|X}(\cdot|x)$. □

Appendix C. Identification

C.1. **Proof of Lemma 2.**

Proof. Our proof is an extension of the copula argument in Darsow et al. (1992). Fix $X = x$. By law of iterated expectation,

$$\begin{aligned}
& \mathbb{P}\left(Y_i = 1; Y_{-i} = a_{-i}|\alpha(X) = \alpha\right) \\
= \; & \mathbb{E}_{U_i}\left[\mathbb{P}\left(Y_i = 1; Y_{-i} = a_{-i}|\alpha(X) = \alpha, U_i\right)\right] \\
= \; & \int_{Q_{U_i}(0)}^{Q_{U_i}(\alpha_i)} \mathbb{P}\left(Y_{-i} = a_{-i}|\alpha(X) = \alpha, U_i = u_i\right) dF_{U_i}(u_i) \\
= \; & \int_0^{\alpha_i} \mathbb{P}\left[Y_{-i} = a_{-i}|\alpha(X) = \alpha, U_i = Q_{U_i}(v_i)\right] dv_i \\
= \; & \int_0^{\alpha_i} \mathbb{P}\left[Y_{-i} = a_{-i}|\alpha_{-i}(X) = \alpha_{-i}, U_i = Q_{U_i}(v_i)\right] dv_i
\end{aligned}$$

where the second equality comes from assumption E and the fact that $\mathbb{P}[Y_i = 1|\alpha(X) = \alpha, U_i \leq Q_{U_i}(\alpha_i)] = 1$ and $\mathbb{P}[Y_i = 1|\alpha(X) = \alpha, U_i > Q_{U_i}(\alpha_i)] = 0$, the third equality from a change–in–variable ($v_i = F_{U_i}(u_i)$) in the integration, and the last step is because conditioning on $\alpha_{-i}(X)$ is equivalent to conditioning on $u_j^*(X)$ for all $j \neq i$, therefore, Y_{-i} is (conditionally) independent of X (and $\alpha_i(X)$ as well) given $\alpha_{-i}(X)$ by assumption E.

Therefore, we have

$$\frac{\partial \mathbb{P}\left(Y_i = 1; Y_{-i} = a_{-i} | \alpha(X) = \alpha\right)}{\partial \alpha_i} = \mathbb{P}\left[Y_{-i} = a_{-i} | \alpha(X) = \alpha, U_i = Q_{U_i}(\alpha_i)\right].$$

Note that $Q_{U_i}(\alpha_i(x)) = u_i^*(x)$. By assumption E, we then have

$$\begin{aligned}
\sigma_{-i}^*(a_{-i}|x, u_i^*(x)) &\equiv \mathbb{P}\left[Y_{-i} = a_{-i} | X = x, U_i = u_i^*(x)\right] \\
&= \mathbb{P}\left[Y_{-i} = a_{-i} | \alpha(X) = \alpha(x), U_i = u_i^*(x)\right] = \frac{\partial \mathbb{P}\left(Y_i = 1; Y_{-i} = a_{-i} | \alpha(X) = \alpha\right)}{\partial \alpha_i}\Big|_{\alpha = \alpha(x)}. \quad \square
\end{aligned}$$

C.2. **Proof of Lemma 3.**

Proof. By the proof in Lemma 1 and assumption ER, we have: for all $x \in \mathscr{S}_X$ such that $\alpha_i(x) \in (0,1)$,

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \sigma_{-i}^*\big(a_{-i} | x, u_i^*(x)\big) = Q_{U_i}(\alpha_i(x)). \tag{11}$$

It follows that

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \mathbb{E}\left[\sigma_{-i}^*\big(a_{-i} | X, u_i^*(X)\big) | X_i = x_i, \alpha_i(X) = \alpha_i(x)\right] = Q_{U_i}(\alpha_i(x)) \tag{12}$$

The difference between (11) and (12) yields

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \times \overline{\sigma}_{-i}^*\big(a_{-i}, x\big) = 0 \tag{13}$$

where $\overline{\sigma}_{-i}^*\big(a_{-i}, x\big) \equiv \sigma_{-i}^*\big(a_{-i} | x, u_i^*(x)\big) - \mathbb{E}\left[\sigma_{-i}^*\big(a_{-i} | X, u_i^*(X)\big) | X_i = x_i, \alpha_i(X) = \alpha_i(x)\right]$.

When x_i is fixed, we can identify $\pi_i(\cdot, x_i)$ as coefficients by varying $\overline{\sigma}_{-i}^*(a_{-i}, x)$ through x_{-i}. Suppose $\mathcal{R}_i(x_i)$ has rank $2^{I-1} - 1$. Because $\sum_{a_{-i} \in \mathcal{A}_{-i}} \overline{\sigma}_{-i}^*\big(a_{-i}, x\big) = 0$, $\pi_i(a_{-i}, x_i)$ equals the same constant for all $a_{-i} \in \mathcal{A}_{-i}$. By (11), we have $\pi_i(\cdot, x_i) = Q_{U_i}(\alpha_i(x))$. Therefore, $\mathscr{S}_{\alpha_i(X)|X_i = x_i}$ has to be a singleton $\{\alpha_i^\dagger\}$.

Next, suppose $\mathcal{R}_i(x_i)$ has rank $2^{I-1} - 2$. Then we can pick a vector $\pi_i^0(\cdot, x_i) \in \mathbb{R}^{2^{I-1}}$ such that $\pi_i^0(a_{-i}, x_i) \neq \pi_i^0(a'_{-i}, x_i)$ for some $a_{-i}, a'_{-i} \in \mathcal{A}_{-i}$, and $\pi_i^0(\cdot, x_i)$ satisfy

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i^0(a_{-i}, x_i) \times \overline{\sigma}_{-i}^*\big(a_{-i}, x\big) = 0.$$

Note that we also have $\sum_{a_{-i} \in \mathcal{A}_{-i}} 1 \times \overline{\sigma}_{-i}^*\big(a_{-i}, x\big) = 0$. By linear algebra, π_i can be written as

$$\pi_i(\cdot, x_i) = c_i(x_i) + p_i(x_i) \times \pi_i^0(\cdot, x_i)$$

where $c_i, p_i : \mathscr{S}_{X_i} \to \mathbb{R}$. Hence, π_i are identified up to location (c_i) and scale (p_i). $\square$

C.3. **Proof of Proposition 5.** For the first half for the proposition, we show by contradiction. It is straightforward that $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ has to be a singleton if $\pi_i(\cdot, x_i)$ is constant on $\mathcal{A}_{-i}$.

We now show the identification of the sign of $\pi_i(a_i, x_i) - \pi_i(a_i^0, x_i)$. Let $x, x' \in \mathscr{S}_{X|X_i=x_i}$, $\alpha(x) = \alpha$, $\alpha(x') = \alpha'$, and w.l.o.g., $\alpha_i' < \alpha_i$. Then

$$\sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i(a_{-i}, x_i)\sigma^*_{-i}\big(a_{-i}|x', Q_{U_i}(\alpha_i)\big) < Q_{U_i}(\alpha_i(x)) = \sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i(a_{-i}, x_i)\sigma^*_{-i}\big(a_{-i}|x, Q_{U_i}(\alpha_i)\big),$$

from which we have

$$p_i(x_i) \times \sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i^0(a_{-i}, x_i) \times \big[\sigma^*_{-i}\big(a_{-i}|x', Q_{U_i}(\alpha_i)\big) - \sigma^*_{-i}\big(a_{-i}|x, Q_{U_i}(\alpha_i)\big)\big] < 0.$$

Thus we identify the sign of $p_i(x_i)$. It follows that the sign of $\pi_i(a_{-i}, x_i) - \pi_i(a'_{-i}, x_i) = p_i(x_i) \times \big[\pi_i^0(a_{-i}, x_i) - \pi_i^0(a'_{-i}, x_i)\big]$ is also identified. □

C.4. **Proof of Proposition 6.**

Proof. By (8) and assumption N, clearly $\pi_i(a_{-i}, \cdot)$ is identified on $\mathbb{C}_i^0$. Hence, it suffices to show that the identification of $\pi_i(a_{-i}, \cdot)$ on $\mathbb{C}_i^t$ implies its identification on $\mathbb{C}_i^{t+1}$. By Definition 2, it suffices to consider $x_i \in \mathbb{C}_i^{t+1}/\mathbb{C}_i^t$.

Suppose that Case (ii) occurs, i.e. $\mathcal{R}_i(x_i)$ has rank $2^{I-1} - 2$ and there exists $x_i' \in \mathbb{C}_i^t$ such that $\mathscr{S}_{\alpha_i(X)|X_i=x_i} \bigcap \mathscr{S}_{\alpha_i(X)|X_i=x_i'} \bigcap (0,1)$ contains at least two different elements $0 < \alpha_i' < \alpha_i < 1$. Let $x, x' \in \mathscr{S}_{X|X_i=x_i}$, $\alpha_i(x) = \alpha_i$ and $\alpha_i(x') = \alpha_i'$. Because $x_i' \in \mathbb{C}_i^t$, then by assumption $\pi_i(\cdot, x_i')$ are identified. Then both $Q_{U_i}(\alpha_i)$ and $Q_{U_i}(\alpha_i')$ are identified by (8). Further, because $\mathcal{R}_i(x_i)$ has rank $2^{I-1} - 2$, then by Lemma 3, $\pi_i(\cdot, x_i)$ is identified up to location and scale, i.e. $\exists\, c_i(x_i), p_i(x_i) \in \mathbb{R}$ and a known vector $\pi_i^0(\cdot, x_i) \in \mathbb{R}^{2^{I-1}}$, such that $\pi_i(\cdot, x_i) = c_i(x) + p_i(x) \times \pi_i^0(\cdot, x_i)$. Moreover, because $\alpha_i, \alpha_i' \in \mathscr{S}_{\alpha_i(X)|X_i=x_i} \bigcap (0,1)$, then we have

$$\sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \times \sigma^*_{-i}(a_{-i}|x, u_i^*(x)) = Q_{U_i}(\alpha_i),$$
$$\sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \times \sigma^*_{-i}(a_{-i}|x', u_i^*(x')) = Q_{U_i}(\alpha_i').$$

It follows that

$$c_i(x_i) + p_i(x_i) \times \sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i^0(a_{-i}, x_i) \times \sigma^*_{-i}(a_{-i}|x, u_i^*(x)) = Q_{U_i}(\alpha_i),$$
$$c_i(x_i) + p_i(x_i) \times \sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i^0(a_{-i}, x_i) \times \sigma^*_{-i}(a_{-i}|x', u_i^*(x')) = Q_{U_i}(\alpha_i').$$

Note that $Q_{U_i}(\alpha_i)$, $Q_{U_i}(\alpha_i')$, $\pi_i^0(\cdot, x_i)$, $\sigma^*_{-i}(a_{-i}|x, u_i^*(x))$ and $\sigma^*_{-i}(a_{-i}|x', u_i^*(x'))$ are all known terms. Because $Q_{U_i}(\alpha_i') < Q_{U_i}(\alpha_i)$, the determinant of the equation system cannot be zero. Then we can identify $c_i(x_i)$ and $p_i(x_i)$ from the above two equations. Therefore, $\pi_i(\cdot, x_i)$ are identified.

Suppose that Case (iii) occurs, i.e. $\mathcal{R}_i(x_i)$ has rank $2^{I-1} - 1$ and there exists $x_i' \in \mathbb{C}_i^t$ such that $\mathscr{S}_{\alpha_i(X)|X_i=x_i} \subseteq \mathscr{S}_{\alpha_i(X)|X_i=x_i'} \bigcap (0,1)$. By Lemma 3, $\pi_i(\cdot, x_i)$ is identified by $Q_{U_i}(\alpha_i)$, which is known since $\mathscr{S}_{\alpha_i(X)|X_i=x_i} \subseteq \mathscr{S}_{\alpha_i(X)|X_i=x_i'} \bigcap (0,1)$. □

Appendix D. Extensions

D.1. **Proof of Lemma 5.**

Proof. First, we construct a structure $\widetilde{S} \in \mathcal{M}_2$ such that (1) $\widetilde{S}$ has the marginal quantile functions $\left(\widetilde{Q}_{U_1}, \cdots, \widetilde{Q}_{U_I}\right)$; (2) $\widetilde{C}_U(\cdot) = C_U(\cdot)$ on $[0,1]^I$; (3) for any $x \in \mathscr{S}_X$, i, and $a_{-i} \in \mathcal{A}_{-i}$, let $\widetilde{\pi}_i(a_{-i}, x) = \widetilde{Q}_{U_i}\left(\mathbb{E}(Y_i|X = x)\right)$. By construction, it is straightforward that assumptions R, M and E are satisfied.

Now it suffices to verify the observational equivalence between $\widetilde{S}$ and S. Fix $x \in \mathscr{S}_X$. Note that in the structure $\widetilde{S}$ there is no strategic effects, then the equilibrium is: $\mathbf{1}\left\{u_i \leq \widetilde{Q}_{U_i}\left(\mathbb{E}(Y_i|X = x)\right)\right\}$ for $i = 1, \cdots, I$. Here we only verify the observational equivalence for action profile $(1, \cdots, 1)$ and the proofs for other action profiles follow similarly:

$$\begin{aligned}\widetilde{\mathbb{P}}(Y_1 = 1; \cdots; Y_I = 1|X = x) &= \widetilde{C}_U\left(\mathbb{E}(Y|X = x)\right) \\ &= C_U\left(\mathbb{E}(Y|X = x)\right) = \mathbb{P}(Y_1 = 1; \cdots; Y_I = 1|X = x). \quad \square\end{aligned}$$

D.2. **Proof of Proposition 7.**

Proof. It is straightforward that $\pi \in \Theta_I(\{Q_{U_i}\}_{i=1}^I, C_U)$. For sharpness, it suffices to show that for any $\tilde{\pi} \in \Theta_I(\{\tilde{Q}_{U_i}\}_{i=1}^I, \tilde{C}_U)$, then $\tilde{S} \equiv (\tilde{\pi}, \{\tilde{Q}_{U_i}\}_{i=1}^I, \tilde{C}_U)$, which belongs to $\mathcal{M}_2$ by the definition of $\Theta_I(\{\tilde{Q}_{U_i}\}_{i=1}^I, \tilde{C}_U)$, is observationally equivalent to the underlying structure $S \equiv (\pi, \{Q_{U_i}\}_{i=1}^I, C_U)$.

Fix $X = x$. It suffices to verify that $\delta^* = \left(\mathbf{1}\left\{u_1 \leq \tilde{Q}_{U_1}(\alpha_1(x))\right\}, \cdots, \mathbf{1}\left\{u_I \leq \tilde{Q}_{U_I}(\alpha_I(x))\right\}\right)$ is a BNE solution for the constructed structure. Because $\tilde{C}_U \in \mathscr{C}_I$ and by the proof for Lemma 2,

$$\tilde{\mathbb{P}}_{\delta^*}\left\{Y_{-i} = a_{-i}|X = x, U_i = \tilde{Q}_{U_i}(\alpha_i(x))\right\} = \sigma^*_{-i}(a_{-i}|x, u_i^*(x)).$$

Then, by the conditions in the definition of $\Theta_I(\{\tilde{Q}_{U_i}\}_{i=1}^I, \tilde{C}_U)$, $\mathbf{1}\left\{u_i \leq \tilde{Q}_{U_i}(\alpha_i(x))\right\}$ is the best response to δ^*_{-i}. Thus δ^* is a BNE. □

Appendix E. An example

The purpose of this example is (i) to verify the assumptions of the paper using primitive conditions, and (ii) to illustrate the identification results for $\mathcal{M}_2$ in Section 4.2.

E.1. **Verifying Assumptions R and M.** Let $I = 2$. We consider signals (U_1, U_2) satisfying the following assumption.

Assumption TD (Type Distribution): *(i)* $(U_1, U_2) \perp X$. *(ii)* $U_1 \perp U_2 | V$ *with* $U_i | V \sim U[0, V]$ *for* $i = 1, 2$ *and* V *having density* $f_V(v) = 3v^2$ *on* $[0, 1]$.

Assumption TD-(i) is Assumption E in the paper. Assumption TD-(ii) implies

$$f_{U_1 U_2 | V}(u_1, u_2 | v) = \frac{1}{v^2} \text{ on } [0, v]^2, \text{ for } v \in [0, 1];$$
$$f_{U_1 U_2}(u_1, u_2) = 3[1 - \max(u_1, u_2)] \text{ on } [0, 1]^2;$$
$$f_{U_i}(u_i) = \frac{3}{2}(1 - u_i^2) \text{ on } [0, 1], \text{ for } i = 1, 2.$$

In particular, the support of (U_1, U_2) is $[0, 1]^2$ and its density is continuous and strictly positive on $[0, 1)^2$. Thus, Assumption R is satisfied.

Next, for $(u_1^*, u_2^*) \in [0, 1]^2$ we have

$$\mathbb{P}(U_{-i} \leq u_{-i}^* | U_i = u_i^*) = \mathbf{1}(u_i^* < u_{-i}^*) \left[1 - \frac{(1 - u_{-i}^*)^2}{1 - u_i^{*2}} \right] + \mathbf{1}(u_i^* \geq u_{-i}^*) \frac{2u_{-i}^*}{1 + u_i^*}$$

In particular, for any $u_{-i}^* \in [0, 1]$, $\mathbb{P}(U_{-i} \geq u_{-i}^* | U_i = u_i^*)$ is continuous and non-decreasing in $u_i^* \in [0, 1]$. Thus, Assumption PRD is satisfied.

For any $x \in \mathscr{S}_X$, let $\Delta_i(x) \equiv \pi_i(1, x) - \pi_i(0, x) \geq 0$ for $i = 1, 2$. By Lemma 6, it follows that Assumption M is satisfied and there exists an m.p.s BNE. That is, there exists at least one pair of equilibrium thresholds $[u_1^*(x), u_2^*(x)]$ satisfying eq. (3) which can be written as

$$\pi_1(0, x) + \Delta_1(x) \times \mathbb{P}[U_2 \leq u_2^*(x) | U_1 = u_1^*(x)] = u_1^*(x);$$
$$\pi_2(0, x) + \Delta_2(x) \times \mathbb{P}[U_1 \leq u_1^*(x) | U_2 = u_2^*(x)] = u_2^*(x).$$

Assumption P0. *For any* $x \in \mathscr{S}_X$, *we have* $\pi_i(0, x) = 0$ *for* $i = 1, 2$.

To simplify the notation, let $\Delta_i = \Delta_i(x)$ and $u_i^* = u_i^*(x)$. By Assumption P0, if $0 \leq u_2^* \leq u_1^* \leq 1$ with $u_2^* < 1$, the preceding system reduces to

$$\Delta_1 \frac{2u_2^*}{1+u_1^*} = u_1^*; \tag{14}$$

$$\Delta_2 \left[1 - \frac{(1-u_1^*)^2}{1-u_2^{*2}}\right] = u_2^*. \tag{15}$$

On the other hand, if $0 \leq u_1^* \leq u_2^* \leq 1$ with $u_1^* < 1$, then the relevant system is obtained from (14) and (15) by switching the indices 1 and 2.

E.2. **Verifying Assumption SC.** Suppose $(\Delta_1, \Delta_2) \in [0,1]^2$. Note that there always exists a trivial equilibrium $(u_1^*, u_2^*) = (0,0)$ that solves the system (14) and (15) thereby confirming the existence of an m.p.s. BNE. Such an equilibrium is not interesting because of the lack of "cooperation" between the two players. Moreover, note that (14) and (15) imply that if $(u_1^*, u_2^*) \neq (0,0)$, we have

$$\begin{aligned} 2\Delta_1\Delta_2 = \frac{u_1^*(1+u_1^*)(1-u_2^{*2})}{1-(1-u_1^*)^2-u_2^{*2}} &= u_1^*(1+u_1^*)\left[1+\frac{(1-u_1^*)^2}{1-(1-u_1^*)^2-u_2^{*2}}\right] \\ &\geq u_1^*(1+u_1^*)\left[1+\frac{(1-u_1^*)^2}{1-(1-u_1^*)^2}\right] = \frac{1+u_1^*}{2-u_1^*} \geq \frac{1}{2}. \end{aligned} \tag{16}$$

A similar argument applies to the region $0 \leq u_1^* \leq u_2^* \leq 1$ with $u_1^* < 1$. Hence, if $\Delta_1\Delta_2 < \frac{1}{4}$, $(u_1^*, u_2^*) = (0,0)$ is the unique equilibrium. Thus, we focus on $\{(\Delta_1, \Delta_2) \in [\frac{1}{4}, 1]^2 : \Delta_1\Delta_2 \geq \frac{1}{4}\}$ to find another equilibrium.

First, note that when $\Delta_1\Delta_2 = \frac{1}{4}$, the above argument also shows that $(u_1^*, u_2^*) = (0,0)$ is the unique equilibrium. Second, consider the case $\Delta_1\Delta_2 > \frac{1}{4}$ and $\Delta_1 \geq \Delta_2$, which implies that $\Delta_1 > 0.5$. We restrict our attention to $0 \leq u_2^* \leq u_1^* \leq 1$ with $u_2^* < 1$ so that (14) and (15) hold. Note that (14) can be written as $u_1^{*2} + u_1^* - 2\Delta_1 u_2^* = 0$, which always has two roots in u_1^*. The non-negative one is

$$u_1^* = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2\Delta_1 u_2^*} \equiv \phi_1(u_2^*) > 0 \quad \text{for} \quad u_2^* \in (0,1] \tag{17}$$

with $\phi_1(0) = 0$. Moreover $\phi_1(1) = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2\Delta_1} < 1$ if $\Delta_1 < 1$ and $\phi_1(1) = 1$ if $\Delta_1 = 1$. Note that $\phi_1(\cdot)$ is strictly increasing and concave on $[0,1]$. Furthermore $\phi_1'(0) = 2\Delta_1 > 1$. Hence, $\phi_1(\cdot)$ intersects the 45^o-line at $2\Delta_1 - 1 \in (0,1]$, and we have $\phi_1(u_2^*) \geq u_2^*$ for $u_2^* \in [0, 2\Delta_1 - 1]$. Figure 1 displays $\phi_1(\cdot)$ for $\Delta_1 = 0.8$.

Regarding (15), it can be written as $(1-u_1^*)^2 = (1-u_2^*/\Delta_2)(1-u_2^{*2})$ since $\Delta_2 > 0$ and $u_2^* < 1$. In particular, this requires $u_2^* \leq \Delta_2$ as the LHS is non-negative. Hence,

$$u_1^* = 1 - \sqrt{(1-\frac{u_2^*}{\Delta_2})(1-u_2^{*2})} \equiv \phi_2(u_2^*) > 0 \quad \text{for } u_2^* \in (0, \Delta_2] \tag{18}$$

with $\phi_2(0) = 0$. Note that $\phi_2(\Delta_2) = 1$. Moreover, $\phi_2(\cdot)$ is strictly increasing and convex on $[0, \Delta_2]$.[23] We also note that $\phi_2'(0) = \frac{1}{2\Delta_2}$. Furthermore, $\phi_2(\cdot)$ intersects the 45^o-line at $2\Delta_1 - 1 \in (0,1]$ when $\Delta_2 = \Delta_1$. Figure 1 displays $\phi_2(\cdot)$ for $\Delta_2 = 0.3125, 0.5, 0.8$.

For any $\Delta_1 \in (\frac{1}{2}, 1)$, we now show that the set of equilibrium thresholds (u_1^*, u_2^*) is the red-highlighted curved segment without the left-hand point in Figure 1. Fix $\Delta_1 \in (\frac{1}{2}, 1)$. Note that the two functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ intersect exactly once at some point $(u_1^*, u_2^*) \in (0,1)^2$ provided $2\Delta_1 > \frac{1}{2\Delta_2}$ (i.e. $\Delta_1\Delta_2 > \frac{1}{4}$). Moreover, $\phi_2(\cdot)$ rotates clockwise as Δ_2 increases in $(\frac{1}{4\Delta_1}, \Delta_1]$.[24] Thus, the intersection point (u_1^*, u_2^*) traces out $\phi_1(\cdot)$ on some nonempty range $(\underline{u}_2^*, \overline{u}_2^*] \subset (0,1)$ as Δ_2 increases in $(\frac{1}{4\Delta_1}, \Delta_1]$. By the inequalities in (16), we have $\underline{u}_2^* = 0$ as the limit of solution u_2^* as $\Delta_2 \downarrow \frac{1}{4\Delta_1}$. Moreover, when $\Delta_2 = \Delta_1$, $u_1^* = u_2^* = 2\Delta_1 - 1$ is the non-trivial solution that solves the equilibrium conditions. Thus, $\overline{u}_2^* = 2\Delta_1 - 1$. By construction, we have $u_1^* \geq u_2^*$, where $u_1^* = \phi_1(u_2^*) = \phi_2(u_2^*)$.

Now, we let Δ_1 smoothly increase in $(\frac{1}{2}, 1)$. The set of equilibrium thresholds for each Δ_1 strictly and smoothly moves up with respect to Δ_1, because $\phi_1(u_2^*)$ continuously rotates up for any given $u_2^* \in [0,1]$ from (17). The upper bound is $\phi_o(u_2^*)$, where

$$\phi_o(t) \equiv -\frac{1}{2} + \sqrt{\frac{1}{4} + 2t}, \quad \text{for } \ t \geq 0$$

which is $\phi_1(\cdot)$ when $\Delta_1 = 1$. When $\Delta_1 = 1$ and $\Delta_2 \in (\frac{1}{4\Delta_1}, \Delta_1) = (\frac{1}{4}, 1)$, the same argument as above shows that the set of equilibrium thresholds (u_1^*, u_2^*) is as displayed in Figure 2 when Δ_2 increases in $(\frac{1}{4}, 1)$. When $(\Delta_1, \Delta_2) = (1,1)$, it can be shown that $(u_1^*, u_2^*) = (1,1)$ is the only equilibrium other than $(0,0)$. Collecting results, we have shown that, when (Δ_1, Δ_2) varies in

[23] The strict monotonicity is straightforward. For the convexity, note that

$$\phi_2''(u_2) = \frac{4(\Delta_2 - 3u_2)(\Delta_2 - u_2)(1-u_2^2) + (1 + 2\Delta_2 u_2 - 3u_2^2)^2}{4\Delta_2^2\left[(1-\frac{u_2}{\Delta_2})(1-u_2^2)\right]^{3/2}}.$$

Thus, it suffices to show

$$4(\Delta_2 - 3u_2)(\Delta_2 - u_2)(1-u_2^2) + (1 + 2\Delta_2 u_2 - 3u_2^2)^2 \geq 0, \ \ \forall\, u_2 \in [0, \Delta_2].$$

Note that $4(\Delta_2 - 3u_2)(\Delta_2 - u_2)(1-u_2^2) + (1 + 2\Delta_2 u_2 - 3u_2^2)^2$ is decreasing in $u_2 \in [0, \Delta_2]$ with derivative

$$-12(\Delta_2 - u_2)(1-u_2^2) \leq 0, \ \ \forall\, u_2 \in [0, \Delta_2].$$

Thus $4(\Delta_2 - 3u_2)(\Delta_2 - u_2)(1-u_2^2) + (1 + 2\Delta_2 u_2 - 3u_2^2)^2 \geq (1-\Delta_2^2)^2 \geq 0$ for all $u_2 \in [0, \Delta_2]$.

[24] This statement follows Milgrom and Shannon (1994) and the fact that (18) is strictly decreasing in Δ_2.

$\{(\Delta_1, \Delta_2) \in [0,1]^2 : \Delta_1\Delta_2 > \frac{1}{4}, \Delta_1 \geq \Delta_2\}$, the corresponding range of equilibrium thresholds is $\{(u_1^*, u_2^*) \in (0,1]^2 : u_2^* \leq u_1^* \leq \phi_o(u_2^*)\}$.

Next, we turn to the second case $\Delta_1\Delta_2 > \frac{1}{4}$ and $\Delta_1 \leq \Delta_2$. By switching the indices 1 and 2, we obtain a similar result. Namely, when (Δ_1, Δ_2) varies in $\{(\Delta_1, \Delta_2) \in [0,1]^2 : \Delta_1\Delta_2 > \frac{1}{4}, \Delta_1 \leq \Delta_2\}$, the corresponding range of equilibrium thresholds is $\{(u_1^*, u_2^*) \in (0,1]^2 : u_1^* \leq u_2^* \leq \phi_o(u_1^*)\}$. Thus, we have established the following lemma.

Lemma 8. *Suppose Assumptions TD and P0 hold. There exists a pair of equilibrium thresholds* $[u_1^*(x), u_2^*(x)]$ *for every pair of payoff differences* $(\Delta_1(x), \Delta_2(x)) \in [0,1]^2$. *The corresponding set of equilibrium threshold pairs is* $\mathscr{S}_{u^*} \equiv \{(u_1^*, u_2^*) \in [0,1]^2 : u_1^* \leq \phi_o(u_2^*), u_2^* \leq \phi_o(u_1^*)\}$.

The shaded lens in Figure 3 represents the corresponding set $\mathscr{S}_{u^*}$ of equilibrium threshold pairs.

As a matter of fact, our preceding argument shows that the equilibrium can be written as $u^*(x) = (\xi(\Delta_1(x), \Delta_2(x)), \xi(\Delta_2(x), \Delta_1(x)))$ for some continuous function $\xi : [0,1]^2 \mapsto [0,1]$ satisfying
(i) $\xi(\Delta_1, \Delta_2) \geq \xi(\Delta_2, \Delta_1)$ if $\Delta_1 \geq \Delta_2$, and $\xi(\Delta, \Delta) = 2\Delta - 1$;
(ii) $\xi(\Delta_1, \Delta_2) > 0$ if $\Delta_1\Delta_2 > \frac{1}{4}$, and $\xi(\Delta_1, \Delta_2) = 0$ if $\Delta_1\Delta_2 \leq \frac{1}{4}$;
(iii) $\xi(\Delta_1, \Delta_2)$ is strictly increasing in both its arguments on $\{(\Delta_1, \Delta_2) \in [0,1]^2 : \Delta_1\Delta_2 \geq \frac{1}{4}\}$.

We can now verify Assumption SC under the following assumption.

Assumption S0. *(i) The pair of payoff differences* $(\Delta_1(X), \Delta_2(X))$ *has support* $[0,1]^2$.

Assumption S0-(i) is a condition on the functional form of the payoff differences and/or the support of X. By Lemma 8, it follows that the pair of equilibrium thresholds $[u_1^*(X), u_2^*(X)]$ has support $\{(u_1^*, u_2^*) \in [0,1]^2 : u_1^* \leq \phi_o(u_2^*), u_2^* \leq \phi_o(u_1^*)\}$. Moreover, we have

$$\alpha_i(x) = \mathbb{P}[U_i \leq u_i^*(x)] = \frac{3}{2}u_i^*(x) - \frac{1}{2}u_i^{*3}(x)$$

which is strictly increasing in $u_i^*(x) \in [0,1]$. It follows that the support $\mathscr{S}_{\alpha(X)}$ of $[\alpha_1(X), \alpha_2(X)]$ is

$$\mathscr{S}_{\alpha(X)} = \{(\alpha_1, \alpha_2) \in [0,1]^2 : Q_{U_1}(\alpha_1) \leq \phi_o[Q_{U_2}(\alpha_2)], Q_{U_2}(\alpha_2) \leq \phi_o[Q_{U_1}(\alpha_1)]\}$$

The shaded lens in Figure 4 represents $\mathscr{S}_{\alpha(X)}$. Thus, Assumption SC is verified.

E.3. **Verifying Assumptions ER, V and N.** From Section 4.2, the copula $C_{U_1U_2}(\alpha_1, \alpha_2)$ is identified on $\mathscr{S}_{\alpha(X)}$ by

$$C(\alpha_1, \alpha_2) = \mathbb{P}[Y_1 = 1, Y_2 = 1 | \alpha_1(X) = \alpha_1, \alpha_2(X) = \alpha_2]$$

where $\alpha_i(X) = \mathbb{P}(Y_i = 1|X)$. By Lemma 2, the marginal equilibrium beliefs are identified by

$$\sigma^*_{-i}(1|x, u^*_i(x)) = \frac{\partial C(\alpha_1(x), \alpha_2(x))}{\partial \alpha_i}, \quad i = 1, 2.$$

Next, to identify the payoffs, we make the following exclusion restrictions.

Assumption E0. *Let $X = (W_0, W_1, W_2)$ and $X_i = (W_0, W_i)$. For $i = 1, 2$ and all $x = (w_0, w_1, w_2)$, the payoff differences satisfy $\Delta_i(x) = \Delta_i(x_i)$ where $x_i = (w_0, w_i)$.*

Thus, W_i is specific to player i while W_0 is common to both players. Under Assumptions P0 and E0, Assumption ER holds. Moreover, $\sigma^*_{-i}(1|X, u^*_i(X)) = \frac{u^*_i(X)}{\Delta_i(X_i)}$ provided $\Delta_i(X_i) > 0$, because of (3) and Assumptions P0 and E0. Therefore,

$$\begin{aligned}
\overline{\Sigma}^*_{-i}(X) &= \begin{pmatrix} \sigma^*_{-i}(1|X, u^*_i(X)) - \mathrm{E}[\sigma^*_{-i}(1|X, u^*_i(X))|X_i, \alpha_i(X)] \\ \sigma^*_{-i}(0|X, u^*_i(X)) - \mathrm{E}[\sigma^*_{-i}(0|X, u^*_i(X))|X_i, \alpha_i(X)] \end{pmatrix} \\
&= \begin{pmatrix} \frac{u^*_i(X)}{\Delta_i(X_i)} - \mathbb{E}\left[\frac{u^*_i(X)}{\Delta_i(X_i)}|X_i, \alpha_i(X)\right] \\ -\frac{u^*_i(X)}{\Delta_i(X_i)} + \mathbb{E}\left[\frac{u^*_i(X)}{\Delta_i(X_i)}|X_i, \alpha_i(X)\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus, the 2×2 covariance matrix $\mathcal{R}_i(x_i)$ of $\overline{\Sigma}^*_{-i}(X)$ conditional on $X_i = x_i$ has rank 0.

Assumption S0. *(ii) The support of $\Delta_i(X_i)$ is monotone in $W_0 \in R$; (iii) The support of W_0, W_1 and W_2 is a cartesian product.*

We now turn to Assumption V and Assumption N. Fix $x_i = (w_0, w_i)$ such that $\Delta_i(x_i) \in (\frac{1}{4}, 1)$. Moreover, we pick $x_{-i} = (w_0, w_{-i})$ and $\tilde{x}_{-i} = (w_0, \tilde{w}_{-i})$ such that $\frac{1}{4} < \Delta_{-i}(x_{-i}) < \Delta_{-i}(\tilde{x}_{-i}) < \Delta_i(x_i)$. Because x_{-i} and $\tilde{x}_{-i}$ can be chosen such that $\Delta_{-i}(x_{-i})$ and $\Delta_{-i}(\tilde{x}_{-i})$ are arbitrarily close under Assumption S0, we have $(u^*_i(x_i, x_{-i}), u^*_{-i}(x_i, \tilde{x}_{-i})) \in S_{(u^*_i(X), u^*_{-i}(X))}$. Therefore, $(\alpha_i(x_i, x_{-i}), \alpha_{-i}(x_i, \tilde{x}_{-i})) \in S_{\alpha(X)}$. See Figure 5. Thus, Assumption V is verified. By Proposition 5, the sign of $\Delta_i(x_i)$ is identified. Furthermore, Assumption S0 implies that there exists some $x^*_i = (w^*_0, w^*_i)$ in the support of X_i such that $\Delta_i(x^*_i) = 1$ and $S_{\Delta_{-i}(X_{-i})|X_i = x^*_i} = [0, 1]$. It follows that $S_{u^*_i(X)|X_i = x^*_i} = [0, 1]$ (see Figure 2), from which Assumption N-(iii) holds. Thus, under the normalization $|\Delta_i(x^*_i)| = 1$ so that Assumption N-(i,ii) holds, we identify $\Delta_i(\cdot)$ on the support of X_i and $Q_{U_i}(\cdot)$ on $(0, 1)$ by Proposition 6 and Corollary 3.

An example satisfying Assumptions E0 and S0 is as follows. Let $\mathcal{S}_X = [-1, 1] \times [0, 1]^2$, and

$$\Delta_1(X_1) = \varphi_1(W_0) \times \mathbf{1}(W_0 \geq 0)[1 - \theta_1(W_1)] + \theta_1(W_1);$$

$$\Delta_2(X_2) = \varphi_2(W_0) \times \mathbf{1}(W_0 < 0)[1 - \theta_2(W_2)] + \theta_2(W_2),$$

where $\theta_1(\cdot)$ and $\theta_2(\cdot)$ are monotone mappings from $[0,1]$ onto $[0,1]$, $\varphi_1(\cdot)$ is a monotone mapping from $[0,1]$ onto $[0,1]$, and $\varphi_2(\cdot)$ is a monotone mapping from $[-1,0]$ onto $[0,1]$. For instance, this is the case when $\theta_i(w_i) = w_i^{k_i}$ and $\varphi_i(w_0) = |w_0|^{k_0}$, where $k_0, k_1, k_2 > 0$. Thus, Assumption N holds for $i = 1$ by setting $w_0^* > 0$ and $w_1^* \in [0,1]$ such that $\varphi_1(w_0^*) = 1$ or $\theta_1(w_1^*) = 1$.

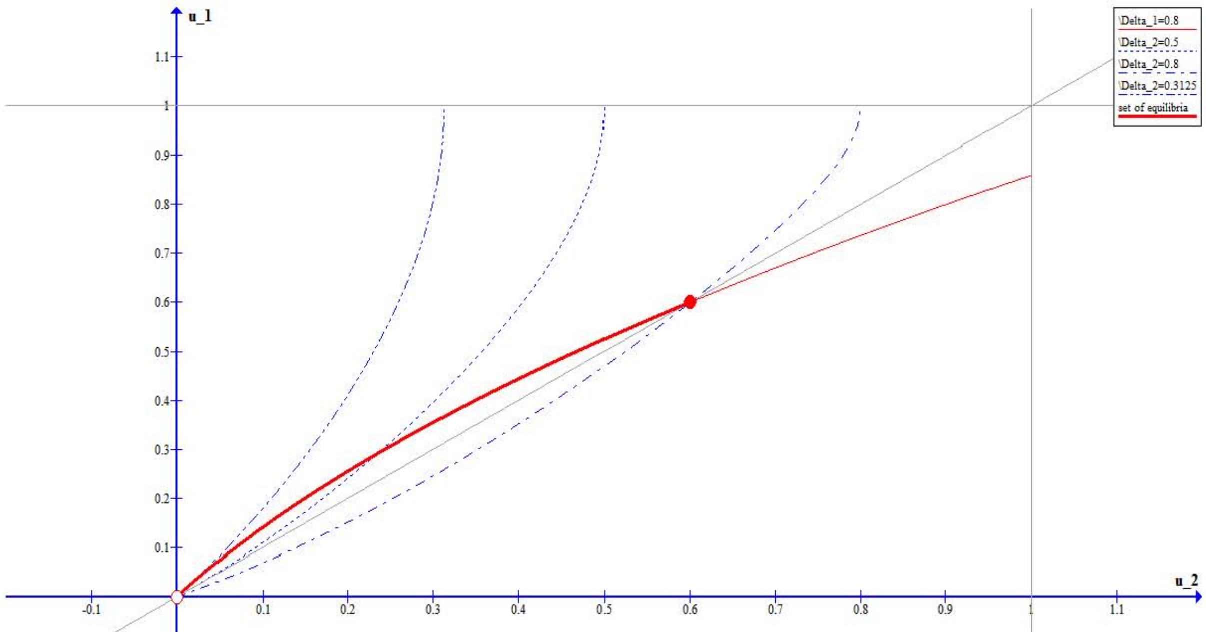

FIGURE 1. $u_1^* = \phi_1(u_2^*)$ for $\Delta_1 = 0.8$ and $u_1^* = \phi_2(u_2^*)$ for $\Delta_2 = 0.3125, 0.5, 0.8$

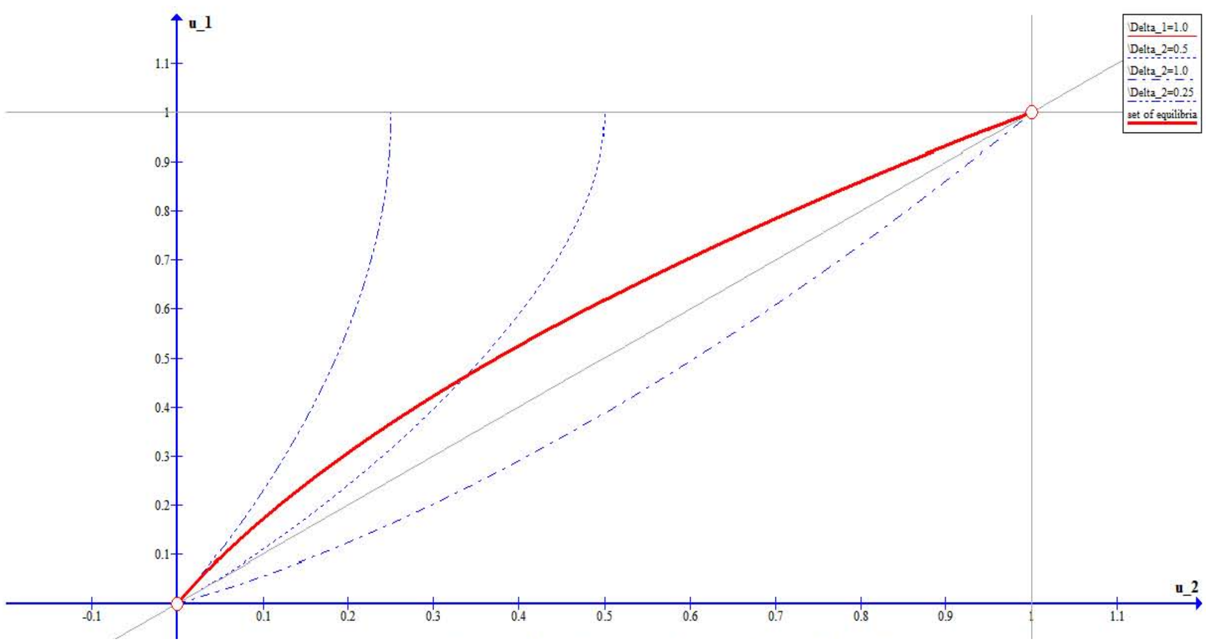

FIGURE 2. $u_1^* = \phi_1(u_2^*)$ for $\Delta_1 = 1$ and $u_1^* = \phi_2(u_2^*)$ for $\Delta_2 = 0.25, 0.5, 1$

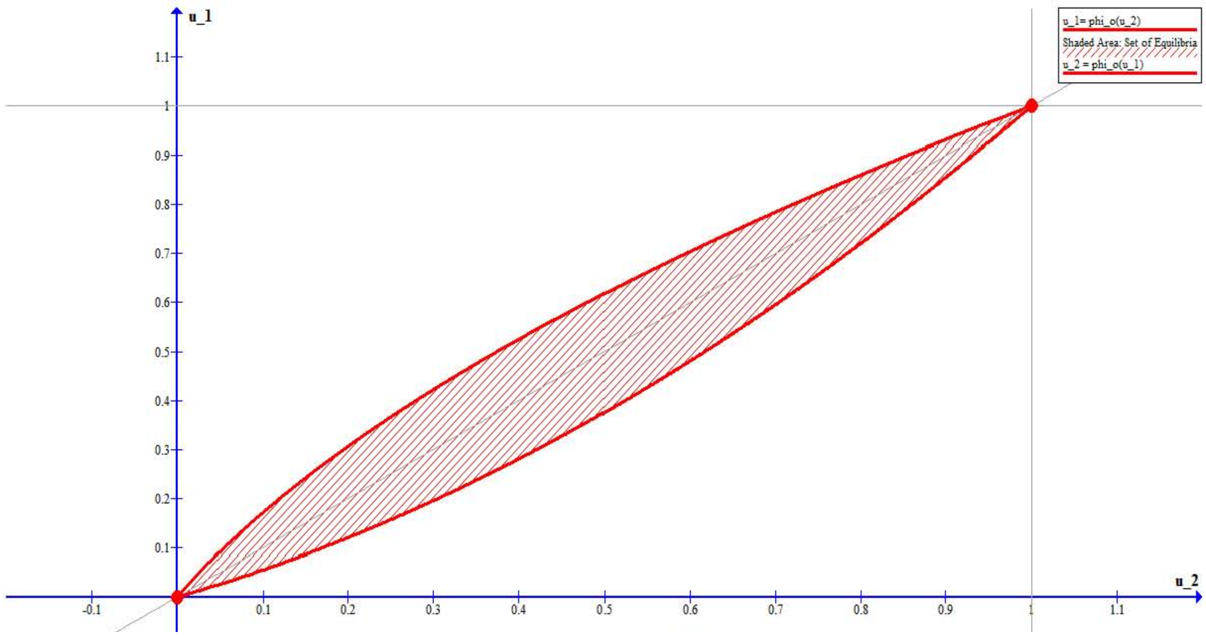

FIGURE 3. Support of $(u_1^*(X), u_2^*(X))$

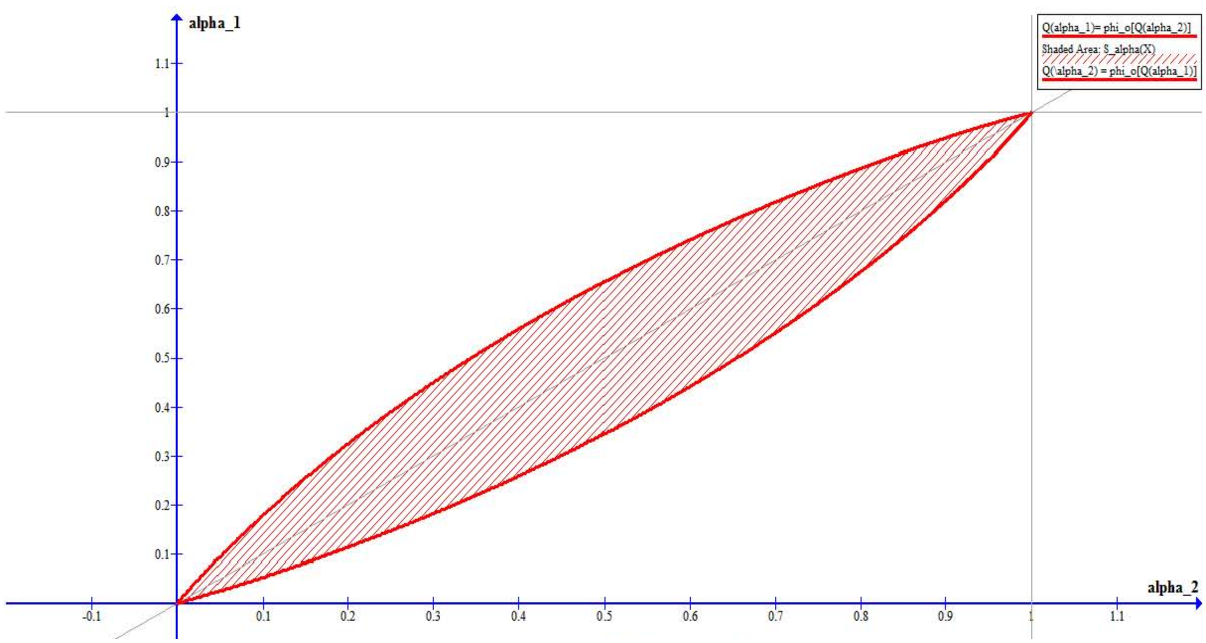

FIGURE 4. Support of $(\alpha_1(X), \alpha_2(X))$

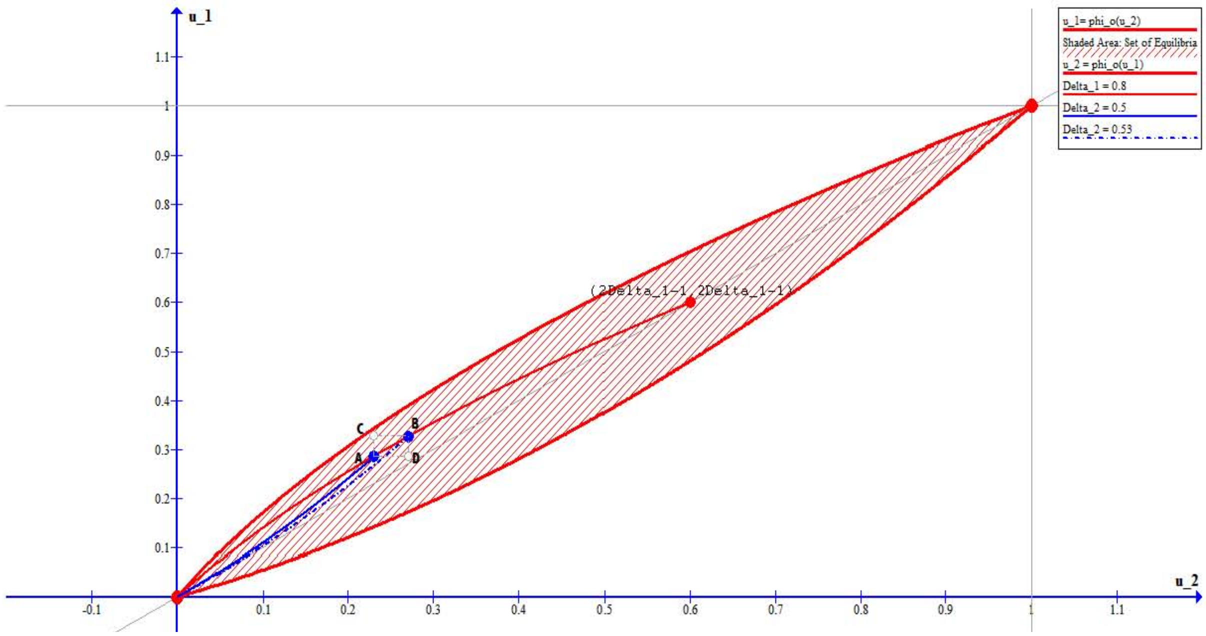

FIGURE 5. Verifying Assumption N-(iii)