

NONPARAMETRIC IDENTIFICATION AND ESTIMATION OF DOUBLE AUCTIONS WITH BARGAINING*

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ABSTRACT. This paper studies the nonparametric identification and estimation of double auctions with one buyer and one seller. This model assumes that both bidders submit their own sealed bids, and the transaction price is determined by a weighted average between the submitted bids when the buyer's offer is higher than the seller's ask. It captures the bargaining process between two parties. Working within this double auction model, we first establish the nonparametric identification of both the buyer's and the seller's private value distributions in two bid data scenarios; from the ideal situation in which all bids are available, to a more realistic setting in which only the transacted bids are available. Specifically, we can identify both private value distributions when all of the bids are observed. However, we can only partially identify the private value distributions on the support with positive (conditional) probability of trade when only the transacted bids are available in the data. Second, we estimate double auctions with bargaining using a two-step procedure that incorporates bias correction. We then show that our value density estimator achieves the same uniform convergence rate as [Guerre, Perrigne, and Vuong \(2000\)](#) for one-sided auctions. Monte Carlo experiments show that, in finite samples, our estimation procedure works well on the whole support and significantly reduces the large bias of the standard estimator without bias correction in both interior and boundary regions.

KEYWORDS: Double auctions, bargaining, nonparametric identification, kernel estimation, boundary correction.

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1. INTRODUCTION

For more than 100 years, trade in the most important field markets for homogeneous goods has been governed primarily by double auction rules (see [Friedman, 1993](#)). With one buyer and one seller, a double auction model captures the nature of bargaining under incomplete information. Applications of such a model range from the settlement of a claim out of court, to union-management negotiations,¹ to the purchase and sale of a used automobile (see [Chatterjee and Samuelson, 1983](#)).

While the theoretical properties of double auctions with bargaining are well established, the theory of identification and estimation in these double auctions is presently very sparse. On the theoretical side, the double auction model with one buyer and one seller has been extensively studied by [Chatterjee and Samuelson \(1983\)](#), [Myerson and Satterthwaite \(1983\)](#), [Leininger, Linhart, and Radner \(1989\)](#), [Satterthwaite and Williams \(1989\)](#), and [Kadan \(2007\)](#), among others. In addition, there is literature which examines the theoretical properties of double auctions with one buyer and one seller in an experimental setting; see, e.g., [Radner and Schotter \(1989\)](#) and [Rapoport and Fuller \(1995\)](#), among others. Nevertheless, there have been few studies of the identification and estimation of double auctions with bargaining from field data. This constrains the corresponding empirical analysis.

Motivated by this gap in the literature, we study the nonparametric identification and estimation of the buyer's and the seller's value distributions in double auctions with bargaining, and obtain the following results: First, in addition to characterizing all the restrictions on the observables (i.e. bid distributions) imposed by the theoretical double auction model with bargaining, we establish point identification of model primitives (i.e. value distributions) from the observables in the case where all bids are observed. In the case when only transacted bids are observed,² we provide a sharp identified set of bidders' value distributions.³ We show that, in the latter case, the conditional distributions of bidders' valuations given positive (conditional) probability of trade are point identified. Second, we propose a (boundary and interior) bias corrected two-step estimator of the buyer's and the seller's value distributions. In a double auction setting, we show that our estimator achieves the same uniform convergence rate as the one-sided auctions provided by [Guerre, Perrigne, and Vuong \(2000\)](#). Third, using Monte Carlo experiments, we show that it is important to implement the bias correction (especially bias correction in the interior of the support) in the two-step estimation of value distributions. In particular, we show that, without bias correction, the statistical inference is almost infeasible, not only on the boundaries, but also in a significant part of the interior.

Our paper builds upon a large body of work which examines nonparametric identification and estimation of one-sided auctions. This work was pioneered by [Guerre, Perrigne, and Vuong \(2000\)](#)

¹ [Treble \(1987, 1990\)](#) obtained documentation of offers and asks in union-management wage negotiations for most UK coalfields over 1893-1914.

² In a transaction, a buyer's bid (or offer) must be no lower than seller's bid (or ask).

³ This result parallels the typical finding that limitations on data observation (such as interval valued data) induce partial identification in nonparametric mean regression and semi-parametric binary regression; see, e.g., [Manski and Tamer \(2002\)](#), [Magnac and Maurin \(2008\)](#), [Wan and Xu \(2015\)](#), among others.

for identification and estimation of first-price auctions, and has been followed by many other papers. Examples include [Li, Perrigne, and Vuong \(2000, 2002\)](#), [Athey and Haile \(2002\)](#), [Haile, Hong, and Shum \(2003\)](#), [Haile and Tamer \(2003\)](#), [Hendricks, Pinkse, and Porter \(2003\)](#), [Li and Zheng \(2009\)](#), [An, Hu, and Shum \(2010\)](#), [Athey, Levin, and Seira \(2011\)](#), [Krasnokutskaya \(2011\)](#), [Tang \(2011\)](#), [Hu, McAdams, and Shum \(2013\)](#), [Gentry and Li \(2014\)](#), among others. For a comprehensive survey, see [Athey and Haile \(2007\)](#). Among these, the identification part of our paper is similar to [Haile and Tamer \(2003\)](#), who obtained bounds on the distribution of valuations by placing two simple assumptions on the relation between valuations and bids without a full characterization of bidding behavior in ascending auctions. Our paper compliments [McAdams \(2008\)](#), who provided upper and lower bounds on the distribution of bidder values in multi-unit auctions, as well as [Tang \(2011\)](#), who bounded counterfactual revenue distributions in auctions with affiliated values. The identification part is also analogous to [Gentry and Li \(2014\)](#), who obtained constructive bounds on model fundamentals which collapse to point identification when available entry variation is continuous in auctions with selective entry. Compared to this research line, however, we consider a different auction setting (namely, double auctions with bargaining) which introduces not only asymmetric information but also asymmetric bidding strategies.⁴

This paper is also closely related to the structural analysis of noncooperative bargaining models. Many papers in this literature recover the model primitives by exploiting the data on the timing and terms of reaching an agreement after sequential bargaining. Complete information examples include [Merlo \(1997\)](#), [Diermeier, Eraslan, and Merlo \(2003\)](#), [Eraslan \(2008\)](#), [Merlo and Tang \(2012, 2015\)](#), and [Simcoe \(2012\)](#), while [Sieg \(2000\)](#), [Watanabe \(2006\)](#), [Merlo, Ortalo-Magne, and Rust \(2009\)](#), are a set of examples which highlight the role of asymmetric information. Our paper, however, uses the data on offers and asks at the beginning of the bargaining process to estimate the initial valuation distributions of both participating parties. Consequently, our work can be viewed as complimentary to this growing literature.

More broadly, we contribute to a third literature on kernel density estimation with boundary correction. In this line of research, the boundary bias can be corrected by several different methods such as the reflection method (e.g. [Silverman, 1986](#)), the boundary kernel approach (e.g. [Gasser and Müller, 1979](#)), the transformation method (e.g. [Wand, Marron, and Ruppert, 1991](#)), the local linear method (e.g. [Cheng, 1997](#), [Cheng, Fan, Marron, et al., 1997](#), [Zhang and Karunamuni, 1998](#)), the nearest internal point approach (e.g. [Imbens and Ridder, 2009](#)), and the reflection of transformed data approach (e.g. [Karunamuni and Zhang, 2008](#), [Zhang, Karunamuni, and Jones, 1999](#)). Among these, [Zhang, Karunamuni, and Jones \(1999\)](#) proposed a generalized reflection method, which involves a reflection of transformed data, and established the pointwise consistency of their estimator. This approach was later improved by [Karunamuni and Zhang \(2008\)](#). In (one-sided) first-price auctions, [Hickman and Hubbard \(2014\)](#) applied their method to correct the boundary bias of the two-step value density estimator which was first proposed by [Guerre, Perrigne, and](#)

⁴ The asymmetry of bidding strategies arises from the fact that the buyer and the seller have different roles in our double auction model.

Vuong (2000). We also adopt the bias correction ideas of Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) to estimate both bid and private value densities in double auctions with bargaining. Relative to these two papers, however, we generalize their density estimation approach so that it can correct both boundary and interior biases which exist in the equilibrium outcome of our double auction model. Furthermore, we establish the uniform consistency of our generalized density estimator on the whole support.

The rest of this paper is organized as follows. In Section 2, we present the sealed-bid double auction model with bargaining and characterize its equilibrium. Section 3 then studies the identification of private value distributions in two different scenarios. In the first scenario, all of the submitted bids can be observed. In contrast, only those bids with successful transactions can be observed in the second scenario. In Section 4, we estimate both the bid and the value densities with bias correction and establish its uniform consistency. Section 5 uses Monte Carlo experiments to illustrate the finite sample performance of our estimator. We briefly discuss the extension of our estimation approach to the case with auction-specific heterogeneity and/or higher order bias reduction in Section 6. Supplementary results are presented in Appendix A, while proofs are collected in Appendix B.

2. THE k -DOUBLE AUCTION MODEL

We consider a k -double auction where a single and indivisible object is auctioned between a buyer and a seller. Each of them simultaneously submits a bid. If the buyer's offer is no lower than the seller's ask, a transaction is made at a price of their weighted average, i.e. at a price $p(B, S) = kB + (1 - k)S$ where k is a constant in $[0, 1]$, B is the buyer's offer, and S is the seller's ask. Otherwise, there is no transaction. The buyer has a value V for the auctioned object, and the seller has a reservation value C . Consequently, the buyer's payoff is $V - p(B, S)$ and the seller's payoff is $p(B, S) - C$ if a trade occurs; their payoffs are both zero otherwise. Each of them does not know her opponent's valuation but only knows that it is drawn from a distribution F_j ($j = C, V$). The distributions F_V, F_C , and the payment rule are all common knowledge between buyer and seller.

We impose the following assumption on the private values and their distributions.

Assumption A. (i) V and C are independent. (ii) F_V is absolutely continuous on the support $[\underline{v}, \bar{v}] \subset \mathbb{R}_+$ with density f_V . F_C is absolutely continuous on the support $[\underline{c}, \bar{c}] \subset \mathbb{R}_+$ with density f_C .

Under Assumption A, the seller's private value is independent of the buyer's, and the value distributions are absolutely continuous on bounded supports. Such an assumption has been adopted by most theoretical papers on double auctions with bargaining; see, e.g., Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989).

We also impose the following restriction on the supports of F_V and F_C .

Assumption B. The supports of F_V and F_C satisfy $\underline{c} < \bar{v}$.

This assumption requires that the buyer's maximum value must be higher than the seller's minimum cost. It rules out the trivial case of $\bar{v} \leq \underline{c}$ in which there is zero probability of trade in any equilibrium. The special cases of such a support condition have been commonly adopted by the theoretical double auction literature; e.g., [Myerson and Satterthwaite \(1983\)](#), [Leininger, Linhart, and Radner \(1989\)](#), and [Satterthwaite and Williams \(1989\)](#).

Denote by $\beta_B : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}_+$ and $\beta_S : [\underline{c}, \bar{c}] \rightarrow \mathbb{R}_+$ the buyer's and the seller's strategies, respectively. Let $b = \beta_B(v)$ denote the bid of a buyer with realized private value v under strategy β_B . Then, the expected profit of the buyer given the seller's strategy is

$$\pi_B(b, v) = \begin{cases} \int_{\underline{s}}^b [v - p(b, s)] dG_S(s) = \int_{\underline{s}}^b [v - kb - (1 - k)s] dG_S(s), & \text{if } b \geq \underline{s}, \\ 0, & \text{if } b < \underline{s}, \end{cases} \quad (2.1)$$

where G_S is the distribution function of the seller's bid and \underline{s} is the lower endpoint of its support. Similarly, let $s = \beta_S(c)$ denote the ask of a seller with realized private reservation value c under strategy β_S . Then, the expected profit of the seller given the buyer's strategy is

$$\pi_S(s, c) = \begin{cases} \int_s^{\bar{b}} [p(b, s) - c] dG_B(b) = \int_s^{\bar{b}} [kb + (1 - k)s - c] dG_B(b), & \text{if } s \leq \bar{b}, \\ 0, & \text{if } s > \bar{b}, \end{cases} \quad (2.2)$$

where G_B is the distribution function of the buyer's bid and \bar{b} is the upper endpoint of its support.

We adopt the Bayesian Nash equilibrium (BNE) concept throughout.

Definition 1 (Best response). *A buyer's strategy β_B is a best response to β_S if for any buyer's strategy $\tilde{\beta}_B : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}_+$ and each value $v \in [\underline{v}, \bar{v}]$, $\pi_B(\beta_B(v), v) \geq \pi_B(\tilde{\beta}_B(v), v)$. The seller's best response is defined in an analogous way.*

Definition 2 (Bayesian Nash equilibrium). *A strategy profile (β_B, β_S) constitutes a Bayesian Nash equilibrium if β_B and β_S are best responses to each other.*

We exclude some irregular equilibria and focus on those which are well-behaved as described in [Chatterjee and Samuelson \(1983\)](#). Precisely, we impose the following restrictions on the equilibrium:

Assumption C (Regular equilibrium). *The equilibrium strategy profile (β_B, β_S) satisfies*

- A1. β_B and β_S are continuous on their whole domains;
- A2. β_B is continuously differentiable with positive derivative on $[\underline{s}, \bar{v}]$ if $\underline{s} < \bar{v}$; β_S is continuously differentiable with positive derivative on $[\underline{c}, \bar{b}]$ if $\underline{c} < \bar{b}$;
- A3. $\beta_B(v) = v$ if $v \leq \underline{s}$; $\beta_S(c) = c$ if $c \geq \bar{b}$.

We say that an equilibrium satisfying Assumption C is regular. Assumption C basically restricts us to strictly monotone and (piecewise) differentiable strategy equilibria which are quite intuitive in bilateral k -double auctions. As demonstrated by [Satterthwaite and Williams \(1989, Theorem 3.2\)](#), there exist a continuum of regular equilibria when $k \in (0, 1)$ and $[\underline{v}, \bar{v}] = [\underline{c}, \bar{c}] = [0, 1]$. Following

most of the empirical game literature, we adopt the following equilibrium selection mechanism when multiple regular equilibria exist:

Assumption D. *In all observed auctions, the buyers and the sellers play the same regular equilibrium.*

Notice that Assumption D is not restrictive when there is a unique regular equilibrium.

The following lemma characterizes some basic properties of the equilibrium strategy profile.

Lemma 1. *For any equilibrium (β_B, β_S) ,*

- (i) *when $v > \underline{v}$, $\beta_B(v) \leq v$ with strict inequality if $k > 0$;*
- (ii) *when $c < \bar{b}$, $\beta_S(c) \geq c$ with strict inequality if $k < 1$.*

Proof. See Appendix B.1. □

Note that the conclusion of Lemma 1 holds for any BNE (i.e., not only for regular BNE). With condition A3 of Assumption C, it implies that, in regular equilibrium, the buyer will never bid higher than her private value and the seller will never bid lower than her private value. Under the special case of $k = 1/2$, [Leininger, Linhart, and Radner \(1989\)](#) constructed a lemma similar to our Lemma 1.

3. NONPARAMETRIC IDENTIFICATION

In this section, we study the nonparametric identification of private value distributions in two cases which differ in the degree of available data. In the first case, econometricians can observe both the transacted bids and the bids where no transaction takes place.⁵ In the second case, econometricians can only observe the transacted bids.

In both cases, we assume that the pricing weight k in the payment rule is known to econometricians. Such an assumption is not restrictive because the value of k can be recovered by using some additional information about the transaction price, given that the transacted bids are observed. For example, when the mean transaction price is observed, the parameter k is determined by $k = \frac{\mathbb{E}(P) - \mathbb{E}(S^*)}{\mathbb{E}(B^*) - \mathbb{E}(S^*)}$ since $\mathbb{E}(P) = k\mathbb{E}(B^*) + (1 - k)\mathbb{E}(S^*)$ where (B^*, S^*) are the transacted bids. Alternatively, when we observe some quantile of the transaction price, k can be identified by exploiting the property that the price distribution function is continuous and monotone in k (see Appendix A.1 for detailed discussion).

3.1. Identification with Perfect Observability of Bid Distribution. We first consider the nonparametric identification of the k -double auction model with bargaining when researchers observe both the parameter k and the distribution of all submitted bids (including the bids that are not transacted).⁶

⁵ We say that a pair of bids (B, S) is transacted if $B \geq S$.

⁶ This observational environment is theoretically interesting and empirically relevant.

As shown in [Chatterjee and Samuelson \(1983\)](#) and [Satterthwaite and Williams \(1989\)](#), a regular equilibrium (β_B, β_S) in a k -double auction with bargaining can be characterized by the following two differential equations for $v \geq \underline{s}$ and $c \leq \bar{b}$,

$$\beta_B^{-1}(\beta_S(c)) = \beta_S(c) + k\beta'_S(c) \frac{F_C(c)}{f_C(c)}, \quad (3.1)$$

$$\beta_S^{-1}(\beta_B(v)) = \beta_B(v) - (1-k)\beta'_B(v) \frac{1 - F_V(v)}{f_V(v)}, \quad (3.2)$$

where $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$ are the inverse bidding strategies.⁷ For buyer with value $v \geq \underline{s}$, the equilibrium bid under strategy β_B is $b = \beta_B(v)$. Let $\tilde{c} = \beta_S^{-1}(b)$. Since strategy β_S is strictly increasing, $G_S(b) = F_C(\beta_S^{-1}(b)) = F_C(\tilde{c})$. Noting that

$$g_S(b) = \frac{f_C(\beta_S^{-1}(b))}{\beta'_S(\beta_S^{-1}(b))} = \frac{f_C(\tilde{c})}{\beta'_S(\tilde{c})}, \quad v = \beta_B^{-1}(b) = \beta_B^{-1}(\beta_S(\tilde{c})),$$

by (3.1), we have

$$v = b + k \frac{G_S(b)}{g_S(b)}. \quad (3.3)$$

Similarly, for seller with value $c \leq \bar{b}$, we have the following condition by (3.2)

$$c = s - (1-k) \frac{1 - G_B(s)}{g_B(s)}. \quad (3.4)$$

Note that (3.3) and (3.4) only hold for $v \geq \underline{s}$ and $c \leq \bar{b}$. In such a case, we have $\Pr(\beta_B(V) \geq \beta_S(C) | V = v) > 0$ when $v > \underline{s}$ and $\Pr(\beta_B(V) \geq \beta_S(C) | C = c) > 0$ when $c < \bar{b}$. In other words, given the private values, both the buyer and the seller expect that trade occurs with positive probability.⁸ For the buyer with value $v < \underline{s}$ or the seller with value $c > \bar{b}$, there will be no transaction under strategy profile (β_B, β_S) . We define functions $\zeta(b, G_S)$ and $\eta(s, G_B)$ as the right-hand sides of (3.3) and (3.4), respectively. That is,

$$\zeta(b, G_S) \equiv b + k \frac{G_S(b)}{g_S(b)}, \quad \underline{s} \leq b \leq \bar{s}, \quad (3.5)$$

$$\eta(s, G_B) \equiv s - (1-k) \frac{1 - G_B(s)}{g_B(s)}, \quad \underline{b} \leq s \leq \bar{b}. \quad (3.6)$$

By definition, it is straightforward that $\zeta(\underline{s}, G_S) = \underline{s}$ and $\eta(\bar{b}, G_B) = \bar{b}$.

We define $\mathcal{P}_{\mathcal{A}}$ as the collection of absolutely continuous probability distributions with support \mathcal{A} . Let G denote the joint distribution of (B, S) . Here, we restrict ourselves to the regular equilibrium strategies which are strictly increasing and (piecewise) differentiable.

Theorem 1. *Under Assumptions C and D, if $G \in \mathcal{P}_{\mathcal{D}}$ is the joint distribution of regular equilibrium bids (B, S) in a sealed-bid k -double auction with some (F_V, F_C) satisfying Assumptions A and B, then*

C1. *The support $\mathcal{D} = [\underline{b}, \bar{b}] \times [\underline{s}, \bar{s}]$ with $\underline{b} \leq \underline{s} < \bar{b} \leq \bar{s}$;*

⁷ When $c = \underline{c}$, (3.1) implies that $\beta_B^{-1}(\underline{s}) = \underline{s}$. Similarly, (3.2) implies that $\beta_S^{-1}(\bar{b}) = \bar{b}$ when $v = \bar{v}$.

⁸ The transaction occurs when $\beta_B(V) \geq \beta_S(C)$.

- C2. $G(b, s) = G_B(b) \cdot G_S(s)$ and $G_B \in \mathcal{P}_{[\underline{b}, \bar{b}]}$, $G_S \in \mathcal{P}_{[\underline{s}, \bar{s}]}$;
 C3. The function $\xi(\cdot, G_S)$ defined in (3.5) is strictly increasing on $[\underline{s}, \bar{b}]$ and its inverse is differentiable on $[\xi(\underline{s}, G_S), \xi(\bar{b}, G_S)]$;
 C4. The function $\eta(\cdot, G_B)$ defined in (3.6) is strictly increasing on $[\underline{s}, \bar{b}]$ and its inverse is differentiable on $[\eta(\underline{s}, G_B), \eta(\bar{b}, G_B)]$;
 C5. For any $\bar{b} \leq b' \leq \bar{s}$ and for any $b \leq \bar{b}$ such that $\xi(b, G_S) \geq b'$,

$$[\xi(b, G_S) - b']G_S(b') - [\xi(b, G_S) - b]G_S(b) + (1 - k) \int_b^{b'} G_S(s) ds \leq 0; \quad (3.7)$$

- C6. For any $\underline{b} \leq s' \leq \underline{s}$ and for any $s \geq \underline{s}$ such that $\eta(s, G_B) \leq s'$,

$$[s' - \eta(s, G_B)][1 - G_B(s')] - [s - \eta(s, G_B)][1 - G_B(s)] + k \int_{s'}^s [1 - G_B(b)] db \leq 0. \quad (3.8)$$

Proof. See Appendix B.2. □

Theorem 1 shows that the theoretical model of a k -double auction with bargaining does impose some restrictions on the joint distribution of observed bids. Together with Theorem 2 which will be shown immediately, these restrictions can be used to establish a formal test of the theory of k -double auction with bargaining. Specifically, condition C1 of Theorem 1 shows that the buyer's minimum (or maximum) bid is not higher than the seller's minimum (or maximum) bid, and the intersection between the buyer's and the seller's bid supports has a non-empty interior. The latter is mainly due to Assumption B about the supports of private value distributions, which implies that there is always positive probability of trade in any regular equilibrium. Condition C2 shows that the buyer's bid is independent of seller's. This independence result is intuitive given that the buyer's value is independent of the seller's. Conditions C3 and C4 say that the functions $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$, which can be regarded as the inverse bidding strategies, are strictly increasing and differentiable on the interval where there is a positive probability of trade. The strict monotonicity property of inverse bidding strategies comes from the fact that the equilibrium strategies are strictly increasing. Conditions C5 and C6 restrict the bid distributions to have small enough probability in the cases where buyer offers less than minimum ask \underline{s} or seller asks more than maximum offer \bar{b} .⁹

The following theorem shows that, under Assumptions C and D, the converse of Theorem 1 is also true.

Theorem 2. *Under Assumptions C and D, if $G \in \mathcal{P}_{\mathcal{D}}$ satisfies C1–C6, then there exists a unique pair of (F_V, F_C) satisfying Assumptions A and B such that G is the joint distribution of some regular equilibrium bids (B, S) in a sealed-bid k -double auction with (F_V, F_C) .*

Proof. See Appendix B.3. □

Theorem 2 is important for several reasons. First, it shows that conditions C1–C6 on the bid distribution G are sufficient to prove the existence of model structure (F_V, F_C) which satisfies

⁹ Otherwise, the buyer with very high private value or the seller with very low reservation value will have incentive to deviate from the given equilibrium strategy.

Assumptions A and B. Second, suppose that the buyer and the seller behave as predicted by the theoretical model of k -double auction with bargaining, Theorem 2 then shows that the private value distributions (F_V, F_C) under which regular equilibrium exists are identified from the joint distribution of observed bids. Third, the inverse bidding strategies, which are equal to $\zeta(\cdot, G_S)$ and $\eta(\cdot, G_B)$ on the trading interval $[\underline{s}, \bar{b}]$, only rely on the knowledge of distribution G as well as the parameter k . Thus, we can avoid solving the linked differential equations (3.1) and (3.2) in order to determine the equilibrium strategy profile (β_B, β_S) . It is worth pointing out that, to identify (F_V, F_C) , Theorem 2 needs all observables to come from the same equilibrium because of the existence of multiple regular equilibria.

Conditions C5 and C6 are less intuitive, and could be difficult to check in some applications. It will be helpful to provide their sufficient conditions which are easy to verify. We are going to show in the following theorem, which can be viewed as a corollary of Theorem 2, that condition C5 will be automatically satisfied if function $\zeta(\cdot, G_S)$ is strictly increasing not only on interval $[\underline{s}, \bar{b}]$ but also on $[\bar{b}, \bar{s}]$ whenever $\bar{s} > \bar{b}$, and that condition C6 will hold if function $\eta(\cdot, G_B)$ is strictly increasing on the entire domain $[\underline{b}, \bar{b}]$.

Theorem 3. *The conclusion of Theorem 2 holds if $G \in \mathcal{P}_{\mathcal{G}}$ satisfies C1–C2 and*

- C7. *The function $\zeta(\cdot, G_S)$ defined in (3.5) is strictly increasing on $[\underline{s}, \bar{s}]$ and its inverse is differentiable on $[\zeta(\underline{s}, G_S), \zeta(\bar{b}, G_S)]$;*
- C8. *The function $\eta(\cdot, G_B)$ defined in (3.6) is strictly increasing on $[\underline{b}, \bar{b}]$ and its inverse is differentiable on $[\eta(\underline{s}, G_B), \eta(\bar{b}, G_B)]$.*

Proof. See Appendix B.4. □

3.2. Identification with Limited Observability of the Bid Distribution. We now discuss the non-parametric identification of the k -double auction model with less data. In particular, we consider the case where econometricians only observe the weight parameter k and the distribution of transacted bids, but never observe the non-transacted bids. Such a case is more empirically realistic than the first one because the bids without transaction are usually not documented in many data sets.

Our identification strategy consists of two steps. In the first step, we identify the bid distribution in an area, namely $[\underline{s}, \bar{b}]^2$, from the distribution of the transacted bids. In the second step, we then find the inverse bidding strategies for the bids in that area so that we can recover the corresponding private values for the buyer and the seller.

Suppose Assumptions A to D hold. Let G_1 denote the joint distribution of the bids located in $[\underline{s}, \bar{b}]^2$ and let G_2 denote the joint distribution of the transacted bids.¹⁰ Notice that G_2 is known by assumption. In the next paragraph, we will show that G_1 can be identified from G_2 .

¹⁰ Precisely, $G_1(b, s) = \Pr(B \leq b, S \leq s \mid (B, S) \in [\underline{s}, \bar{b}]^2)$ and $G_2(b, s) = \Pr(B \leq b, S \leq s \mid \underline{s} \leq S \leq B \leq \bar{b})$.

By the definition of conditional density, the corresponding densities of G_1 and G_2 , namely g_1 and g_2 , are proportional to the density of G , namely g , in their respective supports. Specifically,

$$g_1(b, s) = \frac{g(b, s)}{m}, \quad g_2(b, s) = \frac{g(b, s)}{m'}, \quad (3.9)$$

where $m = \Pr(\underline{s} \leq S \leq \bar{b}, \underline{s} \leq B \leq \bar{b})$ and $m' = \Pr(\underline{s} \leq S \leq B \leq \bar{b})$. By Theorem 1, bids B and S are independent, i.e. $g(b, s) = g_B(b) \cdot g_S(s)$ for any $(b, s) \in [\underline{b}, \bar{b}] \times [\underline{s}, \bar{s}]$. Consequently, the conditional density g_1 can be expressed in terms of g_2 as follows:

$$g_1(b, s) = \begin{cases} \frac{m'}{m} \cdot g_2(b, s) & \text{if } \underline{s} \leq s \leq b \leq \bar{b} \\ \frac{m'}{m} \cdot \frac{g_2(b', s)g_2(b, s')}{g_2(b', s')} & \text{if } \underline{s} \leq b < s \leq \bar{b} \end{cases}$$

where in the latter case b' and s' are chosen such that $\underline{s} \leq s' < b$ and $s < b' \leq \bar{b}$ (see Figure 1).¹¹ For example, we can choose $b' = (\bar{b} + s)/2$ and $s' = (b + \underline{s})/2$. Since $g_2(\cdot, \cdot)$ is identified directly from the observables, the ratio m'/m is then identified by the fact that $\int_{[\underline{s}, \bar{b}]^2} g_1(b, s) db ds = 1$ as

$$\frac{m'}{m} = \left[1 + \int_{\underline{s} \leq b < s \leq \bar{b}} \frac{g_2(b', s)g_2(b, s')}{g_2(b', s')} db ds \right]^{-1},$$

where b' and s' are chosen for each (b, s) in region II of Figure 1 so that (b', s) , (b, s') and (b', s') all locate in region I. Consequently, the conditional density $g_1(\cdot, \cdot)$ is identified on the support of $[\underline{s}, \bar{b}]^2$.

Next, we recover the inverse bidding strategies for the bids in the area $[\underline{s}, \bar{b}]^2$, i.e. regions I and II of Figure 1. Let G_{1B} and G_{1S} denote the buyer's and the seller's marginal bid distributions, respectively, of the joint distribution G_1 , and let g_{1B} and g_{1S} be their densities. In addition, for any $b, s \in [\underline{s}, \bar{b}]$, define

$$\tilde{\zeta}(b, G_{1S}) \equiv b + k \frac{G_{1S}(b)}{g_{1S}(b)}, \quad (3.10)$$

$$\tilde{\eta}(s, G_{1B}) \equiv s - (1 - k) \frac{1 - G_{1B}(s)}{g_{1B}(s)}. \quad (3.11)$$

The following lemma shows how to recover the inverse bidding strategy for bids in regions I and II from the identified G_1 :

Lemma 2. *If $G \in \mathcal{P}_{\mathcal{G}}$ satisfies C1 and C2, then for all $b, s \in [\underline{s}, \bar{b}]$,*

$$\zeta(b, G_S) = \tilde{\zeta}(b, G_{1S}), \quad (3.12)$$

$$\eta(s, G_B) = \tilde{\eta}(s, G_{1B}). \quad (3.13)$$

Proof. See Appendix B.5. □

¹¹ In the latter case, we use the independence property of the joint density g such that $g(b, s) = g_B(b) \cdot g_S(s)$, $g(b', s) = g_B(b') \cdot g_S(s)$, $g(b, s') = g_B(b) \cdot g_S(s')$ and $g(b', s') = g_B(b') \cdot g_S(s')$.

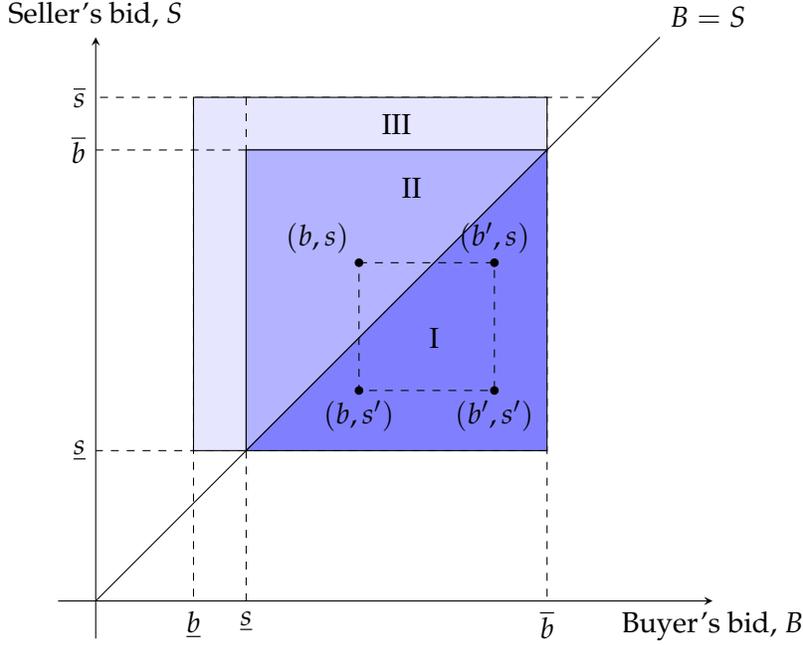


FIGURE 1. Recover G_1 in region II. Here, $\underline{s} \leq s' < b$ and $s < b' \leq \bar{b}$.

Lemma 2 shows that both the buyer's and the seller's inverse bidding strategies are identified in regions I and II of Figure 1, since the conditional distributions of bids in those regions, i.e. G_{1B} and G_{1S} , have been identified in our previous step. Based on this result, we can recover the conditional joint distribution (and hence its marginal distributions) of private values under which the equilibrium bids locate in regions I and II. The specific expressions of those conditional marginal distributions are given by (3.14) of Theorem 4.

The following theorem summarizes the above discussion.

Theorem 4. *Under Assumptions C and D:*

- (i) If $G_2 \in \mathcal{P}_{\mathcal{D}}$ is the joint distribution of transacted bids under some regular equilibrium in a sealed-bid k -double auction with (F_V, F_C) satisfying Assumptions A and B, then
 - D1. The support $\mathcal{D}' = \{(b, s) \mid \underline{s} \leq s \leq b \leq \bar{b}\}$ with $\underline{s} < \bar{b}$;
 - D2. For any $\underline{s} \leq s' \leq s \leq b \leq b' \leq \bar{b}$, the density of G_2 satisfies $g_2(b, s) \cdot g_2(b', s') = g_2(b, s') \cdot g_2(b', s)$;
 - D3. The function $\tilde{\xi}(\cdot, G_{1S})$ defined in (3.10) is strictly increasing on $[\underline{s}, \bar{b}]$ and its inverse is differentiable on $[\tilde{\xi}(\underline{s}, G_{1S}), \tilde{\xi}(\bar{b}, G_{1S})]$;
 - D4. The function $\tilde{\eta}(\cdot, G_{1B})$ defined in (3.11) is strictly increasing on $[\underline{s}, \bar{b}]$ and its inverse is differentiable on $[\tilde{\eta}(\underline{s}, G_{1B}), \tilde{\eta}(\bar{b}, G_{1B})]$.
- (ii) Suppose that Assumptions A and B also hold, and $G_2 \in \mathcal{P}_{\mathcal{D}}$ satisfies D1–D4, then G_2 is the joint distribution of transacted bids under some regular equilibrium in a sealed-bid k -double auction with (F_V, F_C) if and only if (F_V, F_C) satisfies
 - E1. $\underline{c} \leq \underline{s}, \bar{v} \geq \bar{b}$;

E2. For all $(v, c) \in [\underline{s}, \tilde{\xi}(\bar{b}, G_{1S})] \times [\tilde{\eta}(\underline{s}, G_{1B}), \bar{b}]$,¹²

$$\Pr(V \leq v | V \geq \underline{s}) = G_{1B}(\tilde{\xi}^{-1}(v, G_{1S})), \quad \Pr(C \leq c | C \leq \bar{b}) = G_{1S}(\tilde{\eta}^{-1}(c, G_{1B})) \quad (3.14)$$

$$\text{where } \Pr(V \leq v | V \geq \underline{s}) = \frac{F_V(v) - F_V(\underline{s})}{1 - F_V(\underline{s})} \text{ and } \Pr(C \leq c | C \leq \bar{b}) = \frac{F_C(c)}{F_C(\bar{b})};$$

E3. For any $b' \geq \bar{b}$ and for any $b \leq \bar{b}$ such that $\tilde{\xi}(b, G_{1S}) \geq b'$,

$$\begin{aligned} & [\tilde{\xi}(b, G_{1S}) - b']F_C(b') - [\tilde{\xi}(b, G_{1S}) - b]F_C(\tilde{\eta}(b, G_{1B})) \\ & + (1 - k) \left[\int_b^{\bar{b}} F_C(\tilde{\eta}(s, G_{1B})) ds + \int_{\bar{b}}^{b'} F_C(s) ds \right] \leq 0; \quad (3.15) \end{aligned}$$

For any $s' \leq \underline{s}$ and for any $s \geq \underline{s}$ such that $\tilde{\eta}(s, G_{1B}) \leq s'$,

$$\begin{aligned} & [s' - \tilde{\eta}(s, G_{1B})][1 - F_V(s)] - [s - \tilde{\eta}(s, G_{1B})][1 - F_V(\tilde{\xi}(s, G_{1S}))] \\ & + k \left\{ \int_{s'}^{\underline{s}} [1 - F_V(b)] db + \int_{\underline{s}}^s [1 - F_V(\tilde{\xi}(b, G_{1S}))] db \right\} \leq 0. \quad (3.16) \end{aligned}$$

Proof. See Appendix B.6. □

Part (i) of Theorem 4 shows that the conclusion of Theorem 1 carries over to the transacted bids area, i.e. region I of Figure 1, although some (non-transacted) bids cannot be observed now. Specifically, condition D1 says that the support of the distribution of observed (transacted) bids is a triangle in which the buyer's bid is no less than the seller's. Condition D2 means that the multiplication of conditional densities evaluated at (b, s) and (b', s') is the same as the multiplication of conditional densities evaluated at (b, s') and (b', s) as long as these four points are located in the transacted bids area, i.e. region I. Such a condition arises mainly due to the independence of private values. By Lemma 2, conditions D3 and D4 state that both the buyer's and the seller's inverse bidding strategies are strictly increasing and differentiable on the interval of all possible transacted bids values, namely $[\underline{s}, \bar{b}]$.

Part (ii) of Theorem 4 gives the conditions under which the private value distributions rationalize a given distribution of transacted bids. It mainly requires, in the private value interval with positive probability of trade, that both the buyer's and the seller's conditional private value distributions have to generate the corresponding bid distributions when we treat functions $\tilde{\xi}^{-1}$ and $\tilde{\eta}^{-1}$ as buyer's and seller's bidding strategies, respectively. Moreover, part (ii) states that any private value distributions satisfying conditions E1–E3 can rationalize the given distribution of transacted bids, and hence they are observationally equivalent. Although there can be many private value distributions which explain a given distribution of transacted bids, (3.14) shows that the buyer's and the seller's conditional private value distributions are point identified on their value intervals where there is a positive probability of trade.

¹² Notice that we have $\tilde{\xi}(\underline{s}, G_{1S}) = \underline{s}$ and $\tilde{\eta}(\bar{b}, G_{1B}) = \bar{b}$ by the definitions of functions $\tilde{\xi}$ and $\tilde{\eta}$.

4. ESTIMATION

Based on the identification strategy, we provide a nonparametric estimation procedure when all bids can be observed by the researchers, i.e. in the case of Section 3.1. To present the basic ideas, we further assume that all of the observed k -double auctions are homogeneous. Section 6.1 extends our estimation method to allow for auction-specific heterogeneity.

Our estimation procedure extends the two-step estimator proposed by [Guerre, Perrigne, and Vuong \(2000\)](#) for the estimation of sealed-bid first-price auctions: In the first step, a sample of buyers' and sellers' "pseudo private values" is constructed by (3.3) and (3.4), where G_S and G_B are estimated by their empirical distribution functions, and g_S and g_B are estimated by their kernel density estimators with boundary (and interior) bias correction. In the second step, this sample of pseudo private values is used to nonparametrically estimate the densities of buyers' and sellers' private values with boundary bias correction. Notice that, due to the regular equilibrium assumption, a bidder's private value is equal to her bid in the first step if the bidder is a buyer offering less than \underline{s} or if the bidder is a seller asking more than \bar{b} .

It is worth pointing out that a boundary correction is implemented in all kernel density estimators of our two-step procedure.¹³ This is motivated by the fact that boundary bias is worse in double auctions than in first-price auctions. Specifically, as pointed out by [Guerre, Perrigne, and Vuong \(2000\)](#), the estimators of bid density and private value density suffer from boundary bias in the two-step estimation of first-price auctions, since the supports of these two densities are finite. This issue carries over to the double auction setup, and is made worse by the discontinuity of bid densities in the interior of their supports. The interior discontinuity of bid densities occurs because of the strategic asymmetry between the buyer and the seller in double auctions; in a regular equilibrium, the buyer's (or seller's) pseudo private value is recovered via the distribution of her opponent's bid instead of her own by (3.3) (or (3.4)) when her bid is within $[\underline{s}, \bar{b}]$, and is equal to her bid otherwise. This results in the discontinuity of buyer's (or seller's) bid density at interior point \underline{s} (or at interior point \bar{b}). Consequently, the two-step estimator of private value density with boundary and interior bias correction will have much better performance than the one without bias correction (e.g. the one with sample trimming instead) in finite samples. This is confirmed by our Monte Carlo experiments in Section 5.

4.1. Definition of the Estimator. To clarify our idea, we consider n homogeneous k -double auctions. In each auction $i = 1, 2, \dots, n$, there is one buyer with private value V_i and one seller with private value C_i . We observe a sample that consists of all of the buyers' bids $\{B_1, B_2, \dots, B_n\}$ and all of sellers' bids $\{S_1, S_2, \dots, S_n\}$. Let \hat{b} and \hat{s} ($\underline{\hat{s}}$ and $\hat{\bar{s}}$) be the minimum and maximum of the buyers' (sellers') n observed bids.

We first give the general definition of our boundary corrected kernel density estimator. For a random sample $\{X_1, \dots, X_n\}$ that is drawn from distribution F with density f and support $[\underline{x}, \bar{x}]$,¹⁴

¹³ We also need to implement interior bias correction in the estimation of bid densities, i.e. in the first step.

¹⁴ The support $[\underline{x}, \bar{x}]$ is not necessarily bounded.

the boundary corrected kernel density estimator of f on interval $[a_1, a_2] \subseteq [x, \bar{x}]$ is defined as¹⁵

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^n \mathbb{1}(a_1 \leq X_i \leq a_2) \left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x - a_1 + \hat{\gamma}_1(X_i - a_1)}{h}\right) + K\left(\frac{a_2 - x + \hat{\gamma}_2(a_2 - X_i)}{h}\right) \right], \quad (4.1)$$

where

$$\hat{\gamma}_1(u) = u + \hat{d}_1 u^2 + A \hat{d}_1^2 u^3, \quad \hat{\gamma}_2(u) = u + \hat{d}_2 u^2 + A \hat{d}_2^2 u^3,$$

with

$$\begin{aligned} \hat{d}_1 &= \frac{1}{h'} \left\{ \log \left[\frac{1}{nh'} \sum_{i=1}^n \mathbb{1}(a_1 \leq X_i \leq a_2) K\left(\frac{h' - X_i + a_1}{h'}\right) + \frac{1}{n^2} \right] \right. \\ &\quad \left. - \log \left[\max \left(\frac{1}{nh'_0} \sum_{i=1}^n \mathbb{1}(a_1 \leq X_i \leq a_2) K_0\left(\frac{a_1 - X_i}{h'_0}\right), \frac{1}{n^2} \right) \right] \right\}, \\ \hat{d}_2 &= \frac{1}{h'} \left\{ \log \left[\frac{1}{nh'} \sum_{i=1}^n \mathbb{1}(a_1 \leq X_i \leq a_2) K\left(\frac{h' + X_i - a_2}{h'}\right) + \frac{1}{n^2} \right] \right. \\ &\quad \left. - \log \left[\max \left(\frac{1}{nh'_0} \sum_{i=1}^n \mathbb{1}(a_1 \leq X_i \leq a_2) K_0\left(\frac{X_i - a_2}{h'_0}\right), \frac{1}{n^2} \right) \right] \right\}, \end{aligned}$$

$K_0(u) = (6 + 18u + 12u^2) \cdot \mathbb{1}(-1 \leq u \leq 0)$ and

$$h'_0 = \left[\frac{\left(\int_{-1}^1 u^2 K(u) du \right)^2 \left(\int_{-1}^0 K_0^2(u) du \right)}{\left(\int_{-1}^0 u^2 K_0(u) du \right)^2 \left(\int_{-1}^1 K^2(u) du \right)} \right]^{1/5} \cdot h'.$$

Our estimation proceeds as follows: In the first step, we use the observed sample of all bids to estimate the distribution and density functions of the buyers' and sellers' bids by their empirical distribution functions and boundary and interior corrected kernel density estimators, respectively, i.e. by

$$\hat{G}_B(b) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(B_i \leq b), \quad \hat{G}_S(s) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(S_i \leq s),$$

and kernel density estimators \hat{g}_B and \hat{g}_S which are estimated on $[\hat{s}, \hat{b}]$ as shown in (4.1). Specifically, the estimator of the buyers' bid density \hat{g}_B uses kernel function K_B , primary bandwidth h_B , secondary bandwidth h'_B and coefficient $A = A_B$, while the estimator of the sellers' bid density \hat{g}_S uses K_S , h_S , h'_S and $A = A_S$. We then define the buyer's pseudo private value \hat{V}_i corresponding to

¹⁵ We adapt the boundary correction technique proposed by Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) to our double auction setup.

B_i and the seller's pseudo private value \hat{C}_i corresponding to S_i , respectively, as

$$\hat{V}_i = \begin{cases} B_i + k \frac{\hat{G}_S(B_i)}{\hat{g}_S(B_i)} & \text{if } B_i \geq \hat{s}, \\ B_i & \text{otherwise,} \end{cases} \quad \hat{C}_i = \begin{cases} S_i - (1-k) \frac{1 - \hat{G}_B(S_i)}{\hat{g}_B(S_i)} & \text{if } S_i \leq \hat{b}, \\ S_i & \text{otherwise,} \end{cases} \quad (4.2)$$

where $\hat{G}_B(\cdot)$, $\hat{G}_S(\cdot)$, $\hat{g}_B(\cdot)$, and $\hat{g}_S(\cdot)$ are the empirical distribution functions and bias-corrected kernel density estimators defined earlier.

In the second step, we use the pseudo private value samples, $\{\hat{V}_1, \dots, \hat{V}_n\}$ and $\{\hat{C}_1, \dots, \hat{C}_n\}$, to estimate the buyers' and sellers' respective value densities. Specifically, the estimator of the buyers' value density \hat{f}_V is obtained by applying the bias correction approach in (4.1) to the sample of the buyers' pseudo private values on $[\hat{v}, \bar{v}]$, where \hat{v} and \bar{v} are respectively the minimum and maximum of the buyers' pseudo private values, with kernel function K_V , primary bandwidth h_V , secondary bandwidth h'_V , and coefficient $A = A_V$. Similarly, we get the estimator of the sellers' value density \hat{f}_C on interval $[\hat{c}, \bar{c}]$ by the sample of the sellers' pseudo private values with kernel function K_C , primary bandwidth h_C , secondary bandwidth h'_C , and coefficient $A = A_C$.

4.2. Asymptotic Properties. The next assumption concerns the generating process of buyers' and sellers' private values $(V_i, C_i), i = 1, \dots, n$.

Assumption E. $V_i, i = 1, 2, \dots, n$, are independently and identically distributed as F_V with density f_V ; $C_i, i = 1, 2, \dots, n$, are independently and identically distributed as F_C with density f_C .

This assumes that the bidders' private values are independent across auctions. In addition, we impose a smoothness condition on the latent value distributions as follows:

Assumption F. F_V and F_C admit up to $R + 1$ continuous bounded derivatives on $[\underline{v}, \bar{v}]$ and $[\underline{c}, \bar{c}]$, respectively, with $R \geq 1$. In addition, $f_V(v) \geq \alpha_V > 0$ for all $v \in [\underline{v}, \bar{v}]$; $f_C(c) \geq \alpha_C > 0$ for all $c \in [\underline{c}, \bar{c}]$.

Assumption F requires that, on compact supports, the latent value distributions have $R + 1$ continuous derivatives and their density functions are bounded away from zero. As shown in the following lemma, this assumption implies that the generated equilibrium bid distributions will also satisfy a similar smoothness condition.

Lemma 3. *Given Assumption F, the distributions of regular equilibrium bids G_B and G_S satisfy:*

- (i) for any $b \in [\underline{b}, \bar{b}]$ and any $s \in [\underline{s}, \bar{s}]$, $g_B(b) \geq \alpha_B > 0$, $g_S(s) \geq \alpha_S > 0$;
- (ii) G_B and G_S admit up to $R + 1$ continuous bounded derivatives on $[\underline{s}, \bar{b}]$;
- (iii) g_B and g_S admit up to $R + 1$ continuous bounded derivatives on $[\underline{s}, \bar{b}]$.

Proof. See Appendix B.7. □

The striking feature of Lemma 3 is part (iii). It shows that the bid densities are smoother than their corresponding latent value densities. A similar result is obtained by [Guerre, Perrigne, and Vuong \(2000\)](#) in first-price auctions.

We turn to the choice of kernels, bandwidths and other tuning parameters which define our estimator.

Assumption G. (i) The kernels K_B, K_S, K_V, K_C are symmetric with support $[-1, 1]$ and have twice continuous bounded derivatives. (ii) K_B, K_S, K_V and K_C are of order $R + 1, R + 1, R,$ and $R,$ with $R \geq 1$.

Assumption G is standard. The orders of kernels are chosen according to the smoothness of the estimated densities. Specifically, the kernels for bid densities are of order $R + 1$, since the bid densities admit up to $R + 1$ continuous bounded derivatives by Lemma 3. And the kernels for the private value densities are of order R because by Assumption F, the private value densities admit up to R continuous bounded derivatives.

We then give two parallel assumptions which mainly concern the choice of bandwidths.

Assumption H1. The bandwidths h_B, h_S, h_V, h_C are of the form:

$$h_B = \lambda_B \left(\frac{\log n}{n} \right)^{\frac{1}{2R+3}}, \quad h_S = \lambda_S \left(\frac{\log n}{n} \right)^{\frac{1}{2R+3}}, \quad h_V = \lambda_V \left(\frac{\log n}{n} \right)^{\frac{1}{2R+3}}, \quad h_C = \lambda_C \left(\frac{\log n}{n} \right)^{\frac{1}{2R+3}},$$

where the λ 's are positive constants.

Assumption H2. The bandwidths h_B, h_S, h_V, h_C are of the form:

$$h_B = \lambda_B \left(\frac{\log n}{n} \right)^{\frac{1}{5}}, \quad h_S = \lambda_S \left(\frac{\log n}{n} \right)^{\frac{1}{5}}, \quad h_V = \lambda_V \left(\frac{\log n}{n} \right)^{\frac{1}{5}}, \quad h_C = \lambda_C \left(\frac{\log n}{n} \right)^{\frac{1}{5}},$$

where the λ 's are positive constants. The parameters $A_B, A_S, A_V, A_C > 1/3$ and the secondary bandwidths are of the form:

$$h'_B = \tau_B n^{-\frac{1}{5}}, \quad h'_S = \tau_S n^{-\frac{1}{5}}, \quad h'_V = \tau_V n^{-\frac{1}{5}}, \quad h'_C = \tau_C n^{-\frac{1}{5}},$$

where the τ 's are positive constants.

Assumption H1 chooses the primary bandwidths for both the bid and private value densities of order $(\log n/n)^{1/(2R+3)}$. To implement the bias correction technique, we adopt Assumption H2 to choose all primary bandwidths of order $(\log n/n)^{1/5}$ and the secondary bandwidths $h'_B, h'_S, h'_V,$ and h'_C of order $n^{-1/5}$.¹⁶ To extend Assumption H2 for higher order bias reduction, a brief discussion can be found in Section 6.2.

Our main estimation results establish the uniform consistency of the two-step estimator with its rate of convergence. They are built on the following two important lemmas: The first lemma shows the uniform consistency (with rates of convergence) of the first-step nonparametric estimators of the bid densities. The second lemma gives the rate at which the pseudo private values \hat{V}_i and \hat{C}_i converge uniformly to the true private values.

¹⁶ Such choices of secondary bandwidths minimize the mean squared errors of estimating d 's in the transform functions for bias correction.

Lemma 4. (i) Under Assumptions [E](#) to [G](#) and Assumption [H2](#),

$$\sup_{b \in [\underline{b}, \bar{b}]} |\hat{g}_B(b) - g_B(b)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{2}{5}} \right), \quad \sup_{s \in [\underline{s}, \bar{s}]} |\hat{g}_S(s) - g_S(s)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{2}{5}} \right).$$

(ii) Under Assumptions [E](#) to [G](#) and Assumption [H1](#), for any (fixed) closed inner subset \mathcal{C}_g of $[\underline{s}, \bar{s}]$,¹⁷

$$\sup_{b \in \mathcal{C}_g} |\hat{g}_B(b) - g_B(b)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R+1}{2R+3}} \right), \quad \sup_{s \in \mathcal{C}_g} |\hat{g}_S(s) - g_S(s)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R+1}{2R+3}} \right).$$

Proof. See Appendix [B.8](#). □

Lemma 5. (i) Under Assumptions [E](#) to [G](#) and Assumption [H2](#),

$$\sup_i |\hat{V}_i - V_i| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{2}{5}} \right), \quad \sup_i |\hat{C}_i - C_i| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{2}{5}} \right).$$

(ii) Under Assumptions [E](#) to [G](#) and Assumption [H1](#), for any (fixed) closed inner subsets \mathcal{C}_V of $[\underline{v}, \bar{v}]$ and \mathcal{C}_C of $[\underline{c}, \bar{c}]$,

$$\sup_i \mathbb{1}(V_i \in \mathcal{C}_V) |\hat{V}_i - V_i| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R+1}{2R+3}} \right), \quad \sup_i \mathbb{1}(C_i \in \mathcal{C}_C) |\hat{C}_i - C_i| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R+1}{2R+3}} \right).$$

Proof. See Appendix [B.9](#). □

Part (i) of Lemma [4](#) shows that, after bias correction with bandwidth choice outlined in Assumption [H2](#), the kernel density estimators of the bid distributions will uniformly converge in probability to the true densities at a rate of $(\log n/n)^{2/5}$ on their entire supports. Part (i) of Lemma [5](#) then shows that, after the bias correction, all pseudo private values converge uniformly in probability to the true private values at a rate of $(\log n/n)^{2/5}$ under Assumption [H2](#).

Furthermore, part (ii) of Lemmas [4](#) and [5](#) show that, if the primary bandwidths h_B and h_S are of order $(\log n/n)^{1/(2R+3)}$ according to Assumption [H1](#), the bid density estimators and the pseudo private values can have a faster rate of uniform convergence. However, this rate can only be achieved by the bid density estimators on the subsets of the bid interval with positive probability of trade which are strictly bounded away from the support boundaries of bid distributions, and by the pseudo private values corresponding to the observed bids inside these subsets. The rate of uniform convergence in this case, $(\log n/n)^{(R+1)/(2R+3)}$, is the same as the optimal rate obtained by [Guerre, Perrigne, and Vuong \(2000\)](#) for the first-price auctions.

We now give the first main result of the estimation section.

Theorem 5. Under Assumptions [E](#) to [G](#) and Assumption [H1](#), for any (fixed) closed inner subsets \mathcal{C}_V of $[\underline{v}, \bar{v}] \setminus \{\underline{s}\}$ and \mathcal{C}_C of $[\underline{c}, \bar{c}] \setminus \{\bar{b}\}$,

$$\sup_{v \in \mathcal{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R}{2R+3}} \right), \quad \sup_{c \in \mathcal{C}_C} |\hat{f}_C(c) - f_C(c)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R}{2R+3}} \right).$$

¹⁷ We call closed set $\mathcal{A}' \subseteq \mathcal{A}$ a closed inner subset of \mathcal{A} if \mathcal{A}' is also a subset of the interior of \mathcal{A} .

Proof. See Appendix B.10. □

Theorem 5 establishes the uniform consistency of our two-step estimator of the bidders' private value densities. The rate of convergence coincides with the result of [Guerre, Perrigne, and Vuong \(2000\)](#) for the first-price auctions. It is worth pointing out that the convergence rate of the buyers' value density estimator $\hat{f}_V(\cdot)$ can be improved to $(\log n/n)^{R/(2R+1)}$ on the closed inner subsets of $[\underline{v}, \underline{s}]$ when the bandwidth h_V has an order of $(\log n/n)^{1/(2R+1)}$ rather than $(\log n/n)^{1/(2R+3)}$ under Assumption H1. This is due to the fact that the buyer will bid her true value in a regular equilibrium if it is in $[\underline{v}, \underline{s}]$ when $\underline{v} < \underline{s}$ (i.e. we can observe directly her value in this case). A similar conclusion holds for the sellers' value density estimator $\hat{f}_C(\cdot)$ on closed inner subsets of $[\bar{b}, \bar{c}]$.

Theorem 5, however, does not provide the uniform convergence rate of the buyers' (or sellers') value density estimator on a closed inner subset containing \underline{s} (or \bar{b}), although the value density is continuous at this interior point \underline{s} (or \bar{b}). This is caused by the existence of bias in the buyers' (or sellers') value density estimator close to \underline{s} (or \bar{b}). Our next main estimation result addresses this issue by adopting bias correction.

Theorem 6. *Under Assumptions E to G and Assumption H2, for any (fixed) closed inner subsets $\overline{\mathcal{E}}_V$ of $[\underline{v}, \bar{v}]$ and $\overline{\mathcal{E}}_C$ of $[\underline{c}, \bar{c}]$,*

$$\sup_{v \in \overline{\mathcal{E}}_V} |\hat{f}_V(v) - f_V(v)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{1}{5}} \right), \quad \sup_{c \in \overline{\mathcal{E}}_C} |\hat{f}_C(c) - f_C(c)| = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{1}{5}} \right).$$

Proof. See Appendix B.11. □

Theorem 6 establishes the uniform convergence rate of the buyers' (or sellers') value density estimator in any closed inner subsets containing \underline{s} (or \bar{b}). Consequently, we expect that, in comparison to the two-step estimator without bias correction, the one with bias correction will have better finite sample performance close to \underline{s} for the buyers' value density estimator and close to \bar{b} for the sellers'. This is confirmed by our Monte Carlo experiments in the next section. Nevertheless, Theorem 6 does not say anything about the uniform convergence rate on the entire support. The main difficulty comes from the low accuracy in estimation of the boundary points \underline{v} , \bar{v} , \underline{c} and \bar{c} , since they are estimated from the pseudo private values which converge to their true values at a nonparametric rate.

5. MONTE CARLO EXPERIMENTS

To study the finite sample performance of our two-step estimation procedure, we conduct Monte Carlo experiments. We consider two cases of buyers' and sellers' true value distributions and pricing weights. In the first case, both buyers' and sellers' private values are uniformly distributed

on $[0, 1]$. The bidding strategies of the buyer and the seller are given by

$$\beta_B(v) = \begin{cases} \frac{v}{1+k} + \frac{k(1-k)}{2(1+k)}, & \text{if } \frac{1-k}{2} \leq v \leq 1, \\ v, & \text{if } 0 \leq v < \frac{1-k}{2}; \end{cases} \quad \beta_S(c) = \begin{cases} \frac{c}{2-k} + \frac{1-k}{2}, & \text{if } 0 \leq c \leq \frac{2-k}{2}, \\ c, & \text{if } \frac{2-k}{2} < c \leq 1, \end{cases}$$

where k is the pricing weight. Moreover, we set the pricing weight $k = 1/2$ so that the buyer and the seller have equal bargaining power in determining the transaction price. This case has been frequently studied in the theoretical literature (e.g., [Chatterjee and Samuelson, 1983](#)). In the second case, we allow asymmetry between buyers' and sellers' value distributions, and asymmetry between their pricing weights. Specifically, we set the pricing weight to $k = 3/4$, and the true densities of buyers' and sellers' private value distributions to be:

$$f_V(v) = \frac{(8v+12)\sqrt{16v^2-128v+553}-32v^2+80v-105}{(7\sqrt{553}-31)\sqrt{16v^2-128v+553}},$$

$$f_C(c) = \frac{1}{511+\sqrt{73}-1076e^{-3/4}} \left[4 - \frac{8c}{9} + \frac{9+16c}{\sqrt{81+16c^2}} - \frac{2}{9}\sqrt{81+16c^2} + \mathbb{1}(c \geq 3) \frac{(c-3)^3}{3} e^{\frac{3-c}{4}} \right],$$

with identical supports, $[\underline{v}, \bar{v}] = [\underline{c}, \bar{c}] = [0, 6]$.¹⁸ In this case, it can be verified that the buyer's and the seller's bidding strategies given by¹⁹

$$\beta_B(v) = \begin{cases} v, & \text{if } 0 \leq v < 1, \\ \frac{4v+28-\sqrt{16v^2-128v+553}}{11}, & \text{if } 1 \leq v \leq 6; \end{cases}$$

$$\beta_S(c) = \begin{cases} \frac{4c+\sqrt{16c^2+81}}{9}, & \text{if } 0 \leq c \leq 3, \\ c, & \text{if } 3 < c \leq 6, \end{cases}$$

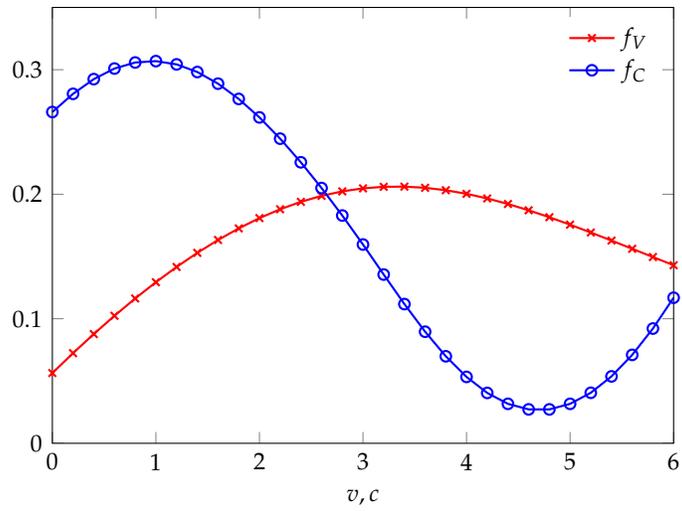
form a regular equilibrium. Figure 2 plots the true value densities, the equilibrium bidding strategies, and the induced bid densities in the second case.

Our Monte Carlo experiment consists of 5000 replications for each case. In each replication, we first randomly generate n buyers' and n sellers' private values from their true value distributions. We then compute the corresponding bids according to the true bidding strategies. Next, we apply our bias-corrected two-step estimation procedure to the generated sample of bids for each replication. In the first step, we estimate the distribution functions and densities of buyers' and sellers' bids using the empirical distribution functions and bias-corrected kernel density estimators, respectively. We then use (4.2) to obtain the buyers' and the sellers' pseudo private values. In the

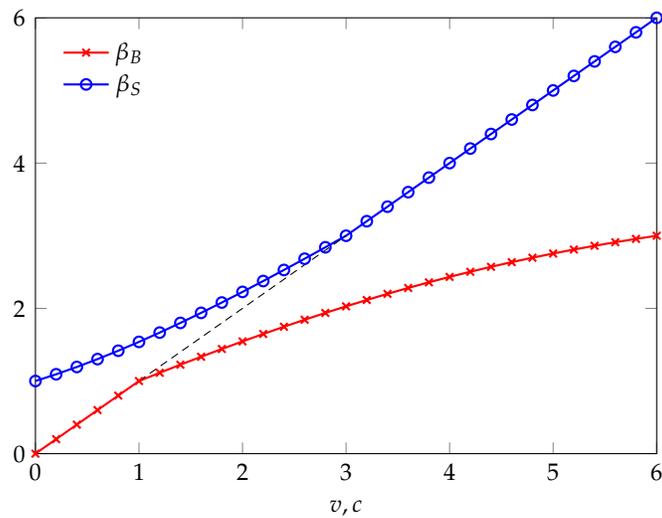
¹⁸ As a matter of fact, we also add some curvature to the true value densities $f_V(\cdot)$ and $f_C(\cdot)$ in this case.

¹⁹ It can also be verified that the corresponding bid densities are

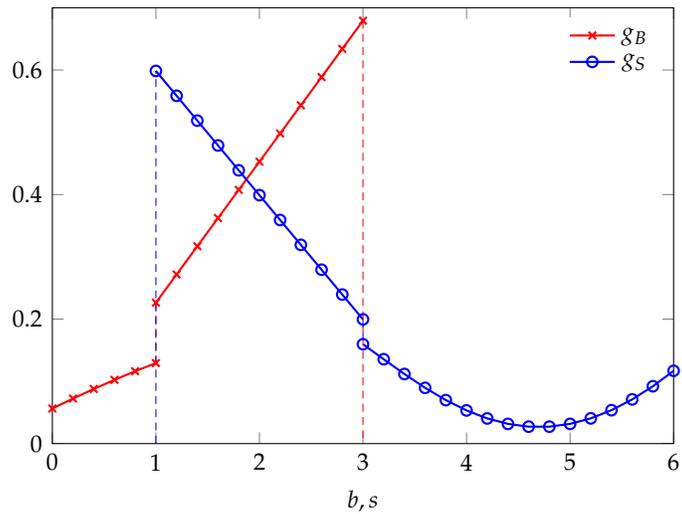
$$g_B(b) = \begin{cases} f_V(b), & \text{if } 0 \leq b < 1, \\ \frac{121b}{28\sqrt{553}-124}, & \text{if } 1 \leq b \leq 3, \\ 0, & \text{otherwise;} \end{cases} \quad g_S(s) = \begin{cases} \frac{36-9s}{2044+4\sqrt{73}-4304e^{-3/4}}, & \text{if } 1 \leq s \leq 3, \\ f_C(s), & \text{if } 3 < s \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$



(A) Buyers' and sellers' private value densities



(B) Equilibrium bidding strategies



(C) Density functions of induced bids

FIGURE 2. True private value densities, equilibrium bidding strategies and bid densities in the second experiment

second step, we use the sample of buyers' and sellers' pseudo private values to estimate buyers' and sellers' value densities by their bias-corrected kernel density estimators.

To satisfy Assumption G on the kernels,²⁰ we choose the triweight kernel for all of $K_B(\cdot)$, $K_S(\cdot)$, $K_V(\cdot)$, and $K_C(\cdot)$, i.e. $K_B(u) = K_S(u) = K_V(u) = K_C(u) = (35/32)(1 - u^2)^3 \cdot \mathbb{1}(-1 \leq u \leq 1)$. We then choose the primary bandwidths h_B, h_S, h_V and h_C according to the rule of optimal global bandwidth (see Silverman, 1986) as

$$h_j = \min \left(n^{-\frac{1}{5}} \hat{\sigma}_j \left[\frac{8\sqrt{\pi} \int_{-1}^1 K_j^2(u) du}{3 \left(\int_{-1}^1 u^2 K_j(u) du \right)^2} \right]^{\frac{1}{5}}, \frac{\hat{r}_j}{2} \right), \quad j = B, S, V, C,$$

where n is the sample size of the observed bids, $\hat{\sigma}_j$ is the estimated standard deviation of observed bids for $j = B, S$ or pseudo private values for $j = V, C$, $K_j(\cdot)$ is the kernel function, and \hat{r}_j is the length of the interval on which the corresponding bid or value density is estimated. In addition, the parameters of bias correction are chosen as follows: all of the coefficients A_B, A_S, A_V and A_C are set at 0.65; each of the secondary bandwidths is equal to its counterpart among the primary bandwidths,²¹ i.e. $h'_j = h_j$ for $j = B, S, V, C$.

Our Monte Carlo results for the first case are summarized in Figure 3. It shows the two-step estimates of value densities with and without bias correction under the sample sizes of $n = 200$ and $n = 1000$, when both buyers' and sellers' private values are uniformly distributed on $[0, 1]$. The true value densities are displayed in solid lines. For each value of $v \in [0, 1]$ (or $c \in [0, 1]$), we plot the mean of the estimates with a dashed line, and the 5th and 95th percentiles with dotted lines. The latter gives the (pointwise) 90% confidence interval for $f_V(v)$ (or $f_C(c)$). Figure 3 shows that our bias-corrected two-step density estimates behave well. First, the true curves fall within their corresponding confidence bands. Second, the mean of the estimates for each density closely matches the true curve. Third, as sample size increases, both the bias and variance of the estimates decrease. Figure 3 also shows that bias correction plays an important role in estimating the value densities in double auctions with bargaining. As shown by Figures 3c, 3d, 3g and 3h, the standard kernel density estimator (without bias correction) has large bias not only at the boundaries but also in an interior area. When the sample size n increases, this bias will not diminish, although the variance will shrink. The appearance of bias in the interior shows that bias correction is necessary to estimate value densities in double auctions with bargaining.

Figure 4 reports the simulation results of the second case under the sample sizes of $n = 200$ and $n = 1000$. Similarly, the true densities, means, and 5th/95th percentiles are respectively displayed in solid lines, dashed lines, and dotted lines. It shows that, with some curvature in the value densities and asymmetry between buyers and sellers, the conclusions in Figure 3 still hold; that

²⁰ Notice that, in both cases, the private value densities $f_V(\cdot)$ and $f_C(\cdot)$ are continuously twice differentiable on the entire support.

²¹ We tried other values of coefficients A_j and secondary bandwidths h'_j , $j = B, S, V, C$, in our experiments, but found that, as long as Assumption H2 holds, the estimates of both buyers' and sellers' value densities are almost the same for different values of A_j and h'_j .

is, (i) the bias-corrected two-step density estimates perform well, and (ii) bias correction plays an important role for estimating the value densities in our double auction model.

6. DISCUSSION

6.1. Auction-Specific Heterogeneity. We now briefly discuss how to generalize our identification and estimation approach to allow for auction-specific heterogeneity.²² Let $X \in \mathbb{R}^d$ be a random vector that characterizes the heterogeneity of auctions. For auctions with $X = x$, let $F_{V|X}(\cdot|x)$ and $F_{C|X}(\cdot|x)$ be the buyers' and sellers' private value distributions, and $G_{B|X}(\cdot|x)$ and $G_{S|X}(\cdot|x)$ be their respective bid distribution functions with densities $g_{B|X}(\cdot|x)$ and $g_{S|X}(\cdot|x)$. Let all of our previous assumptions hold for every x in the support of X wherever it applies. The buyer's and the seller's inverse bidding functions in an auction with characteristic $X = x$ are, respectively,

$$v = \begin{cases} b + \frac{G_{S|X}(b|x)}{g_{S|X}(b|x)}, & \text{if } b \geq \underline{s}(x), \\ b, & \text{otherwise,} \end{cases} \quad c = \begin{cases} s - \frac{1 - G_{B|X}(s|x)}{g_{B|X}(s|x)}, & \text{if } s \leq \bar{b}(x), \\ s, & \text{otherwise,} \end{cases} \quad (6.1)$$

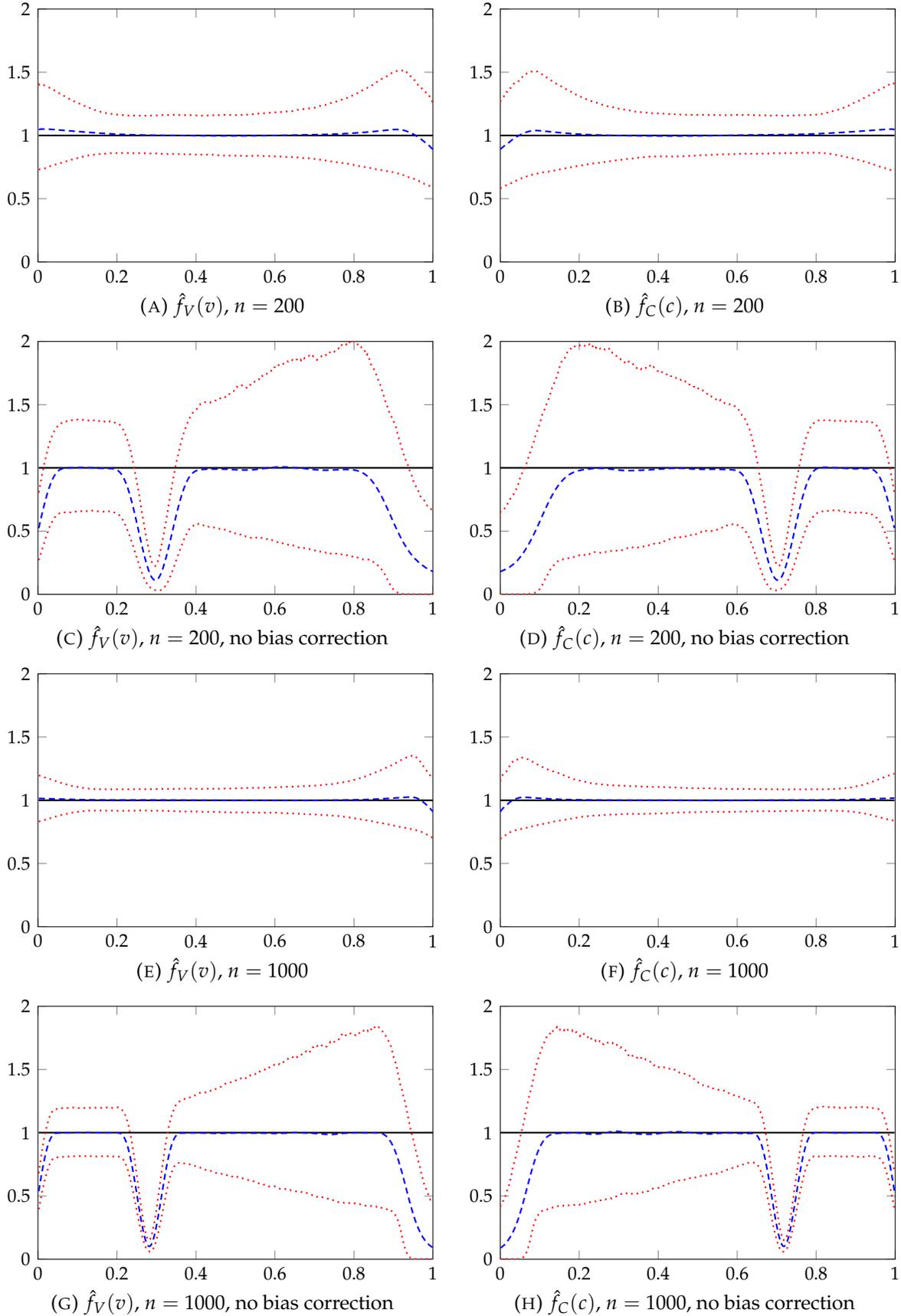
where $\underline{s}(x)$ is the lower bound of the support of $G_{S|X}(\cdot|x)$, $\bar{b}(x)$ is the upper bound of the support of $G_{B|X}(\cdot|x)$.

We can then generalize most of our identification and estimation results to auctions with heterogeneity. Specifically, our identification and model restrictions results (Theorems 1 to 4) still hold as long as the value and bid distributions are simply replaced by the corresponding conditional distributions given X and all relevant conditions hold for every realization of X .

For estimation, our two-step procedure can be generalized to incorporate auction-specific heterogeneity. In the first step, for each auction, we use (6.1) to recover both the buyers' and the sellers' pseudo private values. Notice that, in (6.1), the estimation of conditional bid densities $g_{S|X}$ and $g_{B|X}$ needs to first recover the joint densities g_{SX} and g_{BX} of the bids and the covariates (as well as the marginal density f_X of the covariates), since $g_{S|X}(s|x) = g_{SX}(s,x)/f_X(x)$ and $g_{B|X}(b|x) = g_{BX}(b,x)/f_X(x)$. In the second step, we use the covariate data $\{X_1, \dots, X_n\}$ and pseudo private values recovered previously to estimate the conditional value densities $f_{C|X}$ and $f_{V|X}$. Again, this needs the estimation of joint densities of valuation and covariates f_{CX} and f_{VX} . It is then possible to extend our estimation results in Section 4 to this new two-step estimator. However, the new estimator will suffer the "curse of dimensionality" with the introduction of auction-specific heterogeneity $X \in \mathbb{R}^d$. Moreover, for $d \geq 1$, the (interior and boundary) bias correction in kernel estimation of bid densities g_{SX} and g_{BX} will be an issue in a multi-dimensional scenario.²³ This issue is challenging, in that, to our knowledge, little is known in the existing literature regarding the boundary bias correction of kernel density estimators in a multi-dimensional setting.

²² The existence of auction-specific heterogeneity allows for correlation between the buyer's and the seller's private values. Such correlation, however, exists only through the auction-specific heterogeneity.

²³ Notice that the supports of S and B are finite. In addition, the bid densities can have discontinuity points in the interior of the supports (see Figure 2c).

FIGURE 3. True and estimated densities of private values. $V_i \sim U[0, 1], C_i \sim U[0, 1]$.

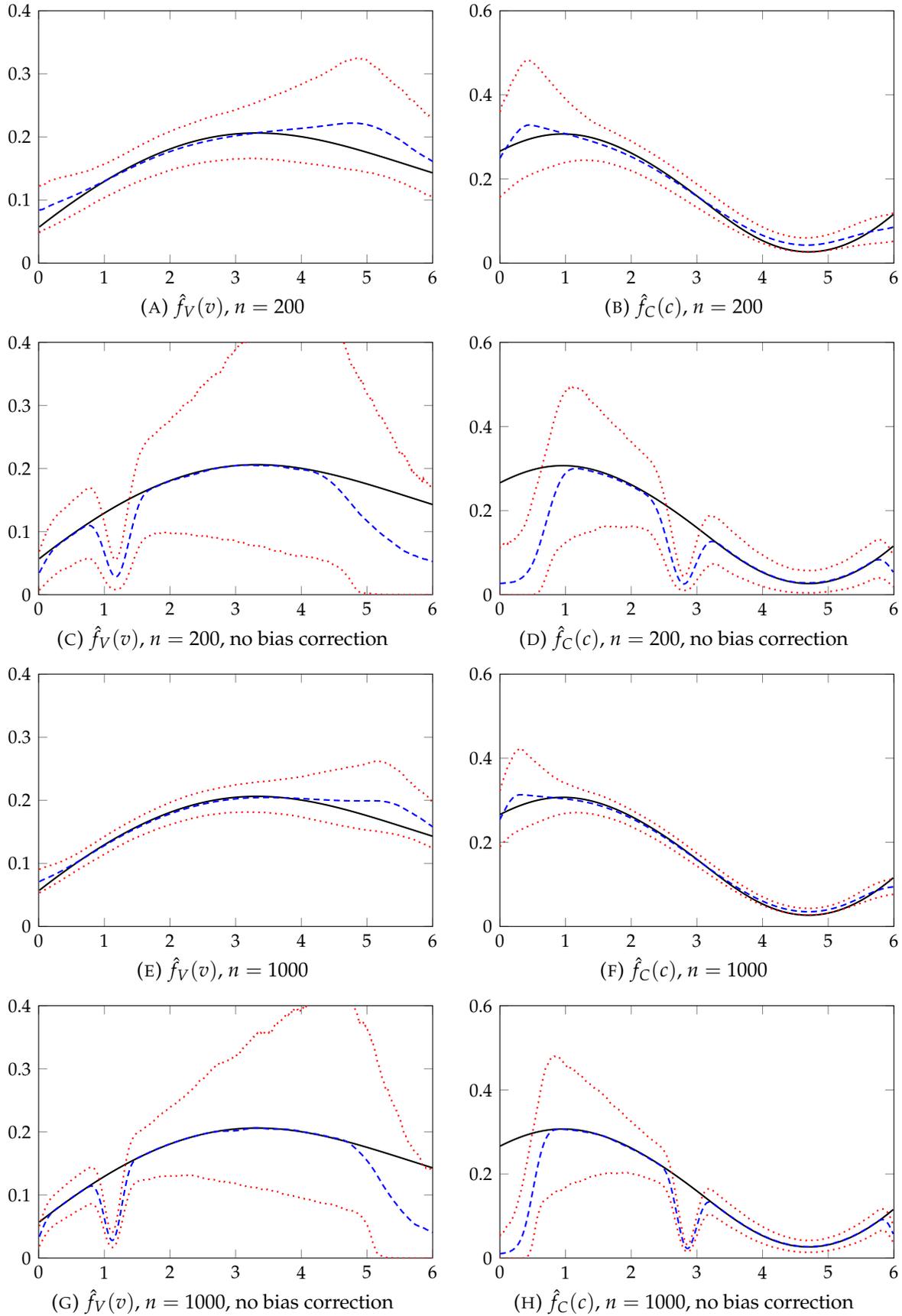


FIGURE 4. True and estimated densities of private values under asymmetry.

6.2. Higher order bias reduction. We can also have higher order boundary (and interior) bias reduction at the cost of more tedious calculations. Due to space limitations, we only illustrate the idea of achieving higher order bias reduction here.

To achieve higher order boundary (and interior) bias reduction, we need to specify both a higher order kernel and a proper functional form for the data transformation. For demonstration purposes, suppose that $\{X_1, X_2, \dots, X_n\}$ is a random sample drawn from a distribution with a density function $f(\cdot)$ and support $[0, \bar{x}]$. To simplify the analysis, we further assume that the density $f(\cdot)$ has a discontinuity point only at 0, i.e. we assume $\lim_{x \rightarrow \bar{x}^-} f(x) = 0$. Denote the transformation function by $\gamma(\cdot)$.²⁴ The (boundary-corrected) kernel density estimator of $f(\cdot)$ with a generalized reflection is given by

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^n \left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x + \gamma(X_i)}{h}\right) \right],$$

where $K(\cdot)$ is a kernel function on support $[-1, 1]$, and h is a bandwidth parameter. Suppose the underlying density $f(\cdot)$ admits up to $R + 1$ continuous bounded derivatives. Let $\omega(\cdot) = f(\gamma^{-1}(\cdot))/\gamma'(\gamma^{-1}(\cdot))$ with $\gamma(\cdot)$ being strictly increasing on $[0, +\infty)$ and $(R + 1)$ -times continuously differentiable. Then, for $x = \rho h$ with $0 \leq \rho \leq 1$, the bias of \hat{f} at x can be obtained as

$$E\hat{f}(x) - f(x) = [\omega(0) - f(0)] \int_{\rho}^1 K(t) dt + \sum_{j=1}^R \frac{W_j}{j!} h^j + O(h^{R+1}), \quad (6.2)$$

where

$$W_j = f^{(j)}(0) \left[\sum_{l=1}^j \binom{j}{l} (-1)^l \rho^{j-l} \int_{-1}^1 t^l K(t) dt \right] + [\omega^{(j)}(0) - (-1)^j f^{(j)}(0)] \int_{\rho}^1 (t - \rho)^j K(t) dt.$$

Consequently, if we choose a kernel $K(\cdot)$ of order $(R + 1)$ and a transformation function $\gamma(\cdot)$ such that (i) $\omega(0) = f(0)$, (ii) $\omega^{(j)}(0) = (-1)^j f^{(j)}(0)$ for all $j = 1, 2, \dots, R$, (iii) $\gamma'(\cdot) > 0$ on $[0, +\infty)$, and (iv) $(R + 1)$ -th derivative of $\gamma(\cdot)$ exists,²⁵ then the boundary bias $E\hat{f}(x) - f(x) = O(h^{R+1})$ for any $x = \rho h$ with $0 \leq \rho \leq 1$. To see this, condition (i) eliminates the first term on the right-hand side of (6.2), and condition (ii) together with $(R + 1)$ -th order kernel $K(\cdot)$ implies $W_j = 0$ for all $j = 1, \dots, R$ which makes the second term on the right-hand side of (6.2) zero. With the bias of order $O(h^{R+1})$ on the boundary, the kernel density estimator $\hat{f}(\cdot)$ with a generalized reflection then converges uniformly to the true density function $f(\cdot)$ at a rate of $O_p\left(h^{R+1} + \sqrt{\log n / (nh)}\right)$ on the entire support $[0, \bar{x}]$.

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²⁴ In Section 4, we follow Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) and employ a cubic transformation function of $\gamma(u) = u + d \cdot u^2 + A \cdot d^2 \cdot u^3$ where d is the derivative of log-density at the boundary point.

²⁵ As a matter of fact, conditions (iii) and (iv) are not essential for the higher order bias reduction.

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APPENDIX A. SUPPLEMENTARY RESULTS

A.1. Identification of Pricing Weight k from Quantiles of Transaction Price. Let $\Psi_k(p) \equiv \Pr(P \leq p)$ be the distribution function of transaction price, where the subscript k indicates the value of this function could also depend on the pricing weight k . Since $\Psi_k(p) = \Pr(kB + (1 - k)S \leq p \mid \underline{s} \leq S \leq B \leq \bar{b})$, for $0 < k < 1$, we have

$$\Psi_k(p) = \begin{cases} \int_{\underline{s}}^p \int_s^{\frac{p-(1-k)s}{k}} g_2(b, s) db ds, & \text{if } p \leq k\bar{b} + (1 - k)\underline{s}, \\ 1 - \int_p^{\bar{b}} \int_{\frac{p-kb}{1-k}}^b g_2(b, s) ds db, & \text{if } p > k\bar{b} + (1 - k)\underline{s}, \end{cases} \quad (\text{A.1})$$

where density function $g_2(b, s)$ is defined by (3.9). When $k = 0$, since $P = S$,

$$\Psi_0(p) = \int_{\underline{s}}^p \int_s^{\bar{b}} g_2(b, s) db ds, \quad (\text{A.2})$$

and similarly, when $k = 1$,

$$\Psi_1(p) = \int_{\underline{s}}^p \int_{\underline{s}}^b g_2(b, s) ds db = \int_{\underline{s}}^p \int_s^p g_2(b, s) db ds. \quad (\text{A.3})$$

In order to establish the conditions on recovering k from the distributions of bids and price, we firstly show the following lemma.

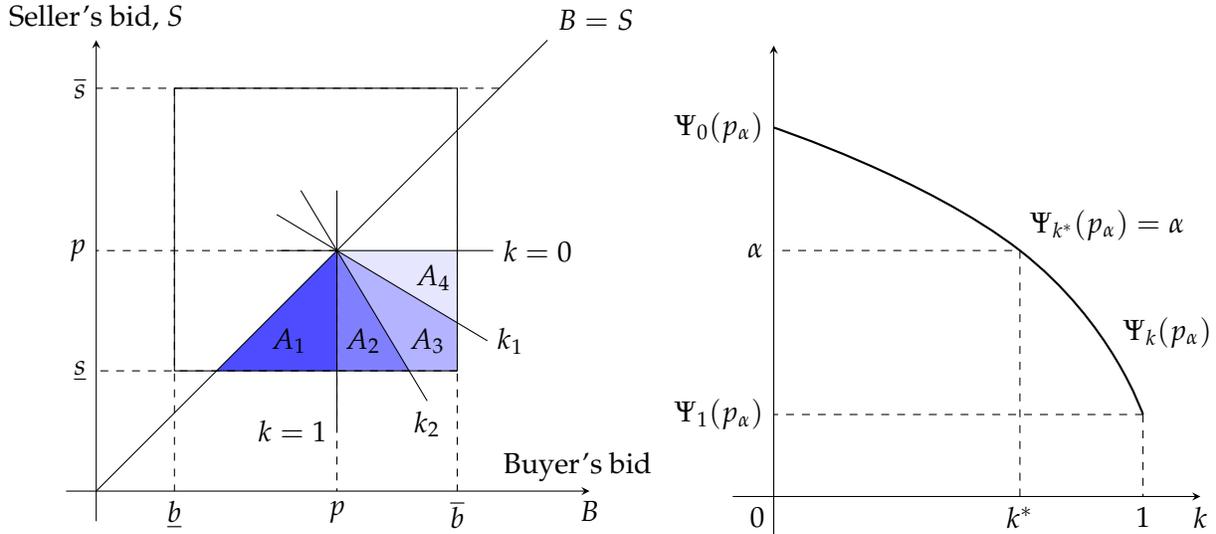
Lemma 6. For any fixed $p \in (\underline{s}, \bar{b})$, $\Psi_k(p)$ is continuous and strictly decreasing in $k \in [0, 1]$.

Proof. See Appendix B.12. □

The intuition behind Lemma 6 is given in Figure 5a. This lemma implies that the distribution function (and hence the quantile function) of transaction price is continuous and strictly monotonic in k . If we know some α th-quantile of the transaction price P , say p_α , such that $\underline{s} < p_\alpha < \bar{b}$ and $\Psi_1(p_\alpha) \leq \alpha \leq \Psi_0(p_\alpha)$, then by Lemma 6, there exists a unique $k^* \in [0, 1]$ such that

$$\Psi_{k^*}(p_\alpha) = \alpha. \quad (\text{A.4})$$

Thus, the value of k can be obtained by solving equation (A.4) for k^* .²⁶ Such an idea is shown by Figure 5b.



(A) Intuition of Lemma 6. Here $0 < k_1 < k_2 < 1$, then
 $\Psi_{k_1}(p) = \iint_{A_1 \cup A_2 \cup A_3} g_2(b, s) db ds$, $\Psi_{k_2}(p) = \iint_{A_1 \cup A_2} g_2(b, s) db ds$.

(B) Recovering k from a price quantile p_α .

FIGURE 5. Identification of pricing weight k from quantiles of transaction price

APPENDIX B. PROOFS

B.1. Proof of Lemma 1. First, we prove that $v > \underline{s}$ implies $\beta_B(v) \leq v$.

When $k = 0$, that is, the transaction price is completely determined by the seller's bid, a buyer with private value $v \geq \underline{s}$ will get

$$\pi_B(b, v) = \int_{\underline{s}}^b (v - s) dG_S(s)$$

²⁶ Notice that, for fixed k and p , $\Psi_k(p)$ is identified from the distribution of transacted bids by (A.1).

from bidding b . Note that the integrand, $v - s$, is strictly decreasing in s , thus

$$\int_{\underline{s}}^b (v - s) dG_S(s) \leq \int_{\underline{s}}^{+\infty} \max\{v - s, 0\} dG_S(s). \quad (\text{B.1})$$

Since $v > \underline{s}$, the equality in (B.1) holds if $b = v$, and the equality holds for all G_S only if $b = v$. This implies that, when $k = 0$, the truthful strategy $\beta_B(v) = v$ is the unique (weakly) dominant strategy for the buyer.

When $k \in (0, 1]$, we shall show that it is better for the buyer with value $v > \underline{s}$ to bid her value v than any bid $b > v$. Since \underline{s} is the lower bound of the support of G_S , $G_S(\underline{s}) = 0$ and $G_S(v) > 0$, then

$$\begin{aligned} \pi_B(v, v) - \pi_B(b, v) &= \int_{\underline{s}}^v [v - kv - (1 - k)s] dG_S(s) - \int_{\underline{s}}^b [v - kb - (1 - k)s] dG_S(s) \\ &= \int_{\underline{s}}^v [v - kv - (1 - k)s] dG_S(s) - \int_{\underline{s}}^v [v - kb - (1 - k)s] dG_S(s) \\ &\quad - \int_v^b [v - kb - (1 - k)s] dG_S(s) \\ &= \int_{\underline{s}}^v k(b - v) dG_S(s) - \int_v^b [v - kb - (1 - k)s] dG_S(s) \\ &= k(b - v)G_S(v) + \int_v^b [kb + (1 - k)s - v] dG_S(s). \end{aligned}$$

Since $b > v$ and $G_S(v) > 0$, the first term is positive and the second term

$$\int_v^b [kb + (1 - k)s - v] dG_S(s) \geq \int_v^b [kb + (1 - k)v - v] dG_S(s) = k(b - v)[G_S(b) - G_S(v)] \geq 0.$$

This completes the proof of $\beta_B(v) \leq v$.

To see that $\beta_B(v) < v$ for $v > \underline{s}$ if $k > 0$, note that by (2.1),

$$\left. \frac{\partial \pi_B(b, v)}{\partial b} \right|_{b=v} = -kG_S(v) < 0.$$

It implies that there exists $\Delta > 0$ small enough such that $\pi_B(v - \Delta, v) > \pi_B(v, v)$, therefore, bidding the true value for the buyer with private value v is no longer optimal, i.e. $\beta_B(v) \neq v$. Since we have already shown that $\beta_B(v) \leq v$, the desired result follows.

In an analogous way, the second conclusion can be proved by showing that truthful bidding strategy is dominant when $k = 1$, and is dominated by some $\tilde{\beta}_S(c) > c$ when $k \in [0, 1)$ and $c < \bar{b}$. \square

B.2. Proof of Theorem 1. Let $\beta_B(\cdot)$ and $\beta_S(\cdot)$ be the respective regular equilibrium bidding strategies of the buyer and the seller that induce the bid distribution G .

By condition A1 of Assumption C, strictly increasing and continuous bidding strategies imply the support of bid distribution is a rectangular region, namely $[\underline{b}, \bar{b}] \times [\underline{s}, \bar{s}]$ with $\underline{b} = \beta_B(\underline{v})$, $\bar{b} = \beta_B(\bar{v})$, $\underline{s} = \beta_S(\underline{c})$ and $\bar{s} = \beta_S(\bar{c})$. To show that $\bar{b} \leq \bar{s}$ and $\underline{b} \leq \underline{s}$, firstly suppose $\bar{b} > \bar{s}$, then any buyer bidding $b > \bar{s}$ will be strictly inferior to just bidding \bar{s} . Because this doesn't make the buyer lose any trades but the expected profit on each trade will increase by lowering the transaction price.

This deviation is contradicted by the assumption that (β_B, β_S) is an equilibrium. Applying similar argument to the seller bidding $s < \underline{b}$, we can prove the second conclusion $\underline{s} \geq \underline{b}$. Then we show that $\underline{s} < \bar{b}$. Suppose not, then: (i) If $\bar{b} \leq \underline{s} < \bar{v}$, the buyer with value \bar{v} will have incentive to bid $\frac{s+\bar{v}}{2}$ instead of \bar{b} , because by bidding $\frac{s+\bar{v}}{2}$ he can get

$$\pi\left(\frac{s+\bar{v}}{2}, \bar{v}\right) = \int_{\underline{s}}^{\frac{s+\bar{v}}{2}} \left[\bar{v} - k\frac{s+\bar{v}}{2} - (1-k)s \right] dG_S(s) = \frac{k}{2}(\bar{v} - \underline{s}) + (1-k) \int_{\underline{s}}^{\frac{s+\bar{v}}{2}} (\bar{v} - s) dG_S(s) > 0$$

while bidding $\bar{b} \leq \underline{s}$ gives him zero expected profit. This contradicts the equilibrium requirement.

(ii) If $\underline{c} < \bar{b} \leq \underline{s}$, then analogous argument can show that bidding $\frac{\bar{b}+\underline{c}}{2}$ is a profitable deviation for the seller with value \underline{c} , which presents a contradiction to the equilibrium condition, too. (iii) If $\bar{b} \leq \underline{c} < \bar{v} \leq \underline{s}$, then condition A3 of Assumption C is contradicted because it requires that $\underline{s} = \underline{c} < \bar{v} = \bar{b}$. From the above, C1 hold.

Because V and C are independent and because $\beta_B(\cdot)$ and $\beta_S(\cdot)$ are deterministic functions, it follows that the bids, $B = \beta_B(V)$ and $S = \beta_S(C)$, are also independent. More precisely, since $\beta_B(\cdot)$ and $\beta_S(\cdot)$ are continuous and strictly increasing, so there exist inverse functions, $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$, which are also continuous and strictly increasing. Thus

$$\begin{aligned} G(b, s) &= \Pr(\beta_B(V) \leq b, \beta_S(C) \leq s) \\ &= \Pr(V \leq \beta_B^{-1}(b), C \leq \beta_S^{-1}(s)) \\ &= \Pr(V \leq \beta_B^{-1}(b)) \Pr(C \leq \beta_S^{-1}(s)) = F_V(\beta_B^{-1}(b)) F_C(\beta_S^{-1}(s)). \end{aligned}$$

Define

$$G_B(b) = F_V(\beta_B^{-1}(b)) \tag{B.2}$$

$$G_S(s) = F_C(\beta_S^{-1}(s)) \tag{B.3}$$

for every $b \in [\underline{b}, \bar{b}]$ and $s \in [\underline{s}, \bar{s}]$. Since $\beta_B^{-1}(\cdot)$ is continuous and strictly increasing on $[\underline{b}, \bar{b}] = [\beta_B(\underline{v}), \beta_B(\bar{v})]$, we have $G_B \in \mathcal{P}_{[\underline{b}, \bar{b}]}$ by (B.2) and the assumption $F_V \in \mathcal{P}_{[\underline{v}, \bar{v}]}$. Similar argument can be applied to show $G_S \in \mathcal{P}_{[\underline{s}, \bar{s}]}$. Now we get C2.

In order to show C3 and C4, note that $G_B(\cdot)$ and $G_S(\cdot)$ defined in (B.2) and (B.3) must be the distributions of observed (equilibrium) bids of the buyer and the seller, respectively. Now, $\beta_B(\cdot)$ and $\beta_S(\cdot)$ must solve the set of first-order differential equations (3.1) and (3.2). Since (3.3) and (3.4) follow from (3.1) and (3.2), then $\beta_B(\cdot)$ and $\beta_S(\cdot)$ must satisfy

$$\zeta(\beta_B(v), G_S) = v, \quad \eta(\beta_S(c), G_B) = c$$

for all $v \geq \underline{s}$ and all $c \leq \bar{b}$. Noting that $\underline{s} = \beta_S(\underline{c})$ and $\bar{b} = \beta_B(\bar{v})$ and making the change of variable $v = \beta_B^{-1}(b)$ and $c = \beta_S^{-1}(s)$, we obtain

$$\zeta(b, G_S) = \beta_B^{-1}(b) \tag{B.4}$$

$$\eta(s, G_B) = \beta_S^{-1}(s) \tag{B.5}$$

for all $b, s \in [\underline{s}, \bar{b}]$. By condition A1 of Assumption C, both $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$ are strictly increasing, and by condition A3 of Assumption C, $\beta_B(\cdot)$ is differentiable on $[\underline{s}, \bar{v}]$ and so is $\beta_S(\cdot)$ on $[\underline{c}, \bar{b}]$. Thus C3 and C4 follow from the fact that $\zeta(\underline{s}, G_S) = \underline{s}$ by (3.3), $\eta(\bar{b}, G_B) = \bar{b}$ by (3.4), and $\bar{v} = \beta_B^{-1}(\bar{b}) = \zeta(\bar{b}, G_S)$, $\underline{c} = \beta_S^{-1}(\underline{s}) = \eta(\underline{s}, G_B)$.

It is remained to show C5 and C6. Given $b \leq \bar{b}$, for buyer with private value v such that $\beta_B(v) = b$, bidding any $b' \in [\bar{b}, \bar{s}]$ should not give him greater profit than bidding b because β_B is the equilibrium bidding strategy for the buyer. That is,

$$\begin{aligned} 0 &\geq \pi_B(b', v) - \pi_B(b, v) = \int_{\underline{s}}^{b'} [v - kb' - (1-k)s] dG_S(s) - \int_{\underline{s}}^b [v - kb - (1-k)s] dG_S(s) \\ &= v[G_S(b') - G_S(b)] - kb'G_S(b') + kbG_S(b) - (1-k) \int_b^{b'} s dG_S(s) \\ &= k(v - b')G_S(b') - k(v - b)G_S(b) \\ &\quad + (1-k) \left[(v - b')G_S(b') - (v - b)G_S(b) + \int_b^{b'} G_S(s) ds \right] \\ &= (v - b')G_S(b') - (v - b)G_S(b) + (1-k) \int_b^{b'} G_S(s) ds. \end{aligned}$$

Because $v = \beta_B^{-1}(b) = \zeta(b, G_S)$ by (B.4), replacing v by $\zeta(b, G_S)$ in the above inequality will yield (3.7). Similarly, for seller with private value c such that $\beta_S(c) = s \geq \underline{s}$, using the argument that any deviation of bidding $s' \in [b, \underline{s}]$ would not be profitable, we can show that (3.8) must hold. This completes the proof of C6 and the theorem. \square

B.3. Proof of Theorem 2. To show the sufficiency of C1–C4, define

$$F_V(v) = \begin{cases} G_B(v) & \text{if } v < \underline{s} \\ G_B(\zeta^{-1}(v, G_S)) & \text{if } \underline{s} \leq v \leq \zeta(\bar{b}, G_S) \\ 1 & \text{if } v > \zeta(\bar{b}, G_S) \end{cases} \quad (\text{B.6})$$

$$F_C(c) = \begin{cases} 0 & \text{if } c < \eta(\underline{s}, G_B) \\ G_S(\eta^{-1}(c, G_B)) & \text{if } \eta(\underline{s}, G_B) \leq c \leq \bar{b} \\ G_S(c) & \text{if } c > \bar{b} \end{cases} \quad (\text{B.7})$$

and

$$\underline{v} = \underline{b}, \quad \bar{v} = \zeta(\bar{b}, G_S), \quad \underline{c} = \eta(\underline{s}, G_B), \quad \bar{c} = \bar{s}.$$

Condition C1 guarantees the functions $\zeta(\cdot, G_S)$ in (3.3) and $\eta(\cdot, G_S)$ in (3.4) are well-defined. Since \underline{b} is the lower endpoint of the support of G_B , so for all $v \leq \underline{v} = \underline{b}$, $F_V(v) = 0$, and by definition, $F_V(v) = 1$ for all $v > \bar{v} = \zeta(\bar{b}, G_S)$. Moreover, because $F_V(\bar{v}) = G_B(\zeta^{-1}(\zeta(\bar{b}, G_S), G_S)) = G_B(\bar{b}) = 1$, $F_V(\underline{s}) = G_B(\zeta^{-1}(\zeta(\underline{s}, G_S), G_S)) = G_B(\underline{s})$, G_B is continuous and strictly increasing on $[\underline{b}, \bar{b}]$ by C2, and $\zeta^{-1}(\cdot, G_S)$ is continuous and strictly increasing on $[\zeta(\underline{s}, G_S), \zeta(\bar{b}, G_S)]$ by C3. Then $F_V(\cdot)$ defined by (B.6) is continuous and strictly increasing on $[\underline{b}, \zeta(\bar{b}, G_S)] = [\underline{v}, \bar{v}]$. Therefore F_V is a valid

absolutely continuous distribution with support $[\underline{v}, \bar{v}]$, i.e. $F_V \in \mathcal{P}_{[\underline{v}, \bar{v}]}$ as required. We can also show $F_C \in \mathcal{P}_{[\underline{c}, \bar{c}]}$ in similar way.

We shall show that the distributions F_V and F_C of buyer's and seller's respective private values can rationalize G in a sealed-bid k -double auction, i.e. $G_B(b) = F_V(\beta_B^{-1}(b))$ on $[\underline{b}, \bar{b}]$ and $G_S(s) = F_C(\beta_S^{-1}(s))$ on $[\underline{s}, \bar{s}]$ for some regular equilibrium profile (β_B, β_S) . By construction of F_V and F_C , we have

$$\begin{aligned} G_B(b) &= F_V(b)\mathbb{1}(\underline{b} \leq b < \underline{s}) + F_V(\xi(b, G_S))\mathbb{1}(\underline{s} \leq b \leq \bar{b}) \\ &= F_V\left(b\mathbb{1}(\underline{b} \leq b < \underline{s}) + \xi(b, G_S)\mathbb{1}(\underline{s} \leq b \leq \bar{b})\right) \end{aligned}$$

for $b \in [\underline{b}, \bar{b}]$ and

$$\begin{aligned} G_S(s) &= F_C(\eta(s, G_B))\mathbb{1}(\underline{s} \leq s \leq \bar{b}) + F_C(s)\mathbb{1}(\bar{b} < s \leq \bar{s}) \\ &= F_C\left(\eta(s, G_B)\mathbb{1}(\underline{s} \leq s \leq \bar{b}) + s\mathbb{1}(\bar{b} < s \leq \bar{s})\right) \end{aligned}$$

for $s \in [\underline{s}, \bar{s}]$, where $\mathbb{1}(\cdot)$ is the indicator function. Define

$$\begin{aligned} \xi_*(b, G_S) &\equiv b\mathbb{1}(\underline{b} \leq b < \underline{s}) + \xi(b, G_S)\mathbb{1}(\underline{s} \leq b \leq \bar{b}), \\ \eta_*(s, G_B) &\equiv \eta(s, G_B)\mathbb{1}(\underline{s} \leq s \leq \bar{b}) + s\mathbb{1}(\bar{b} < s \leq \bar{s}), \end{aligned}$$

then by C3 and C4, $\xi_*(\cdot, G_S)$ is continuous and strictly increasing on $[\underline{b}, \bar{b}]$ and so is $\eta_*(\cdot, G_B)$ on $[\underline{s}, \bar{s}]$. Define bidding strategies

$$\beta_B(v) = \begin{cases} v & \text{if } \underline{v} \leq v \leq \underline{s} \\ \xi_*^{-1}(v, G_S) & \text{if } \underline{s} < v \leq \bar{v} \end{cases} \quad (\text{B.8})$$

$$\beta_S(c) = \begin{cases} \eta_*^{-1}(c, G_B) & \text{if } \underline{c} \leq c < \bar{b} \\ c & \text{if } \bar{b} \leq c \leq \bar{c} \end{cases} \quad (\text{B.9})$$

so that $\beta_B(\cdot) = \xi_*^{-1}(\cdot, G_S)$ and $\beta_S(\cdot) = \eta_*^{-1}(\cdot, G_B)$. By construction of these strategies, A1–A3 in Assumption C are satisfied, and also, $G_B(b) = F_V(\beta_B^{-1}(b))$ and $G_S(s) = F_C(\beta_S^{-1}(s))$ so that G is the induced bid distribution for (F_V, F_C) defined in (B.6) and (B.7) by the strategy profile (β_B, β_S) defined above. Thus it is remained to show (β_B, β_S) is indeed an equilibrium. We show that the optimal bid for the buyer with private value v is $\beta_B(v)$. A similar argument shows that β_S is optimal for the seller.

Obviously, if $v \leq \underline{s}$, then the buyer cannot make an advantageous trade and bidding $\beta_B(v) = v$ achieves zero as her greatest possible expected profit. Suppose $v > \underline{s}$, since G_S is the induced seller's bid distribution, then for bid $b \in [\underline{s}, \bar{b}]$, by (2.1) we obtain

$$\begin{aligned} \frac{\partial \pi_B(b, s)}{\partial b} &= -kG_S(b) + (v - kb)g_S(b) - (1 - k)bg_S(b) \\ &= g_S(b) \left[v - \left(b + k \frac{G_S(b)}{g_S(b)} \right) \right] = g_S(b) [v - \xi(b, G_S)]. \end{aligned}$$

Because $g_S(b)$ is positive, the monotonicity of $\xi(\cdot, G_S)$ by C3 implies that $\partial\pi_B(b, v)/\partial b > 0$ for all $b < \xi^{-1}(v, G_S)$ and $\partial\pi_B(b, v)/\partial b < 0$ for all $b > \xi^{-1}(v, G_S)$. Therefore, $b = \xi^{-1}(v, G_S) = \beta_B(v)$ is the unique maximizer of the buyer's expected profit in $[\underline{s}, \bar{b}]$. Now we show that the buyer would not want to choose bid within $[\bar{b}, \bar{s}]$, either. Recall that we have already shown that C5 is equivalent to $\pi_B(b', v) \leq \pi_B(b, v)$ for any $v \geq \bar{b}$ and any $b' \in [\bar{b}, \bar{s}]$ when $b = \xi^{-1}(v, G_S) = \beta_B(v)$ in the proof of Theorem 1, this is established straightforwardly because choosing a bid within $[\bar{b}, \bar{s}]$ is profitable only for the buyer with private value $v \geq \bar{b}$. Finally, given \bar{s} is the highest seller's bid, any buyer's bid greater than \bar{s} will be dominated by \bar{s} . This completes the proof of sufficiency.

From the proof of Theorem 1, we know that $\xi(\cdot, G_S) = \beta_B^{-1}(\cdot)$ and $\eta(\cdot, G_B) = \beta_S^{-1}(\cdot)$ on $[\underline{s}, \bar{b}]$ when $F_V(\cdot)$ and $F_C(\cdot)$ exist. Since $F_V(\cdot) = G_B(\beta_B(\cdot))$ and $F_C(\cdot) = G_S(\beta_S(\cdot))$, then $F_V(\cdot) = G_B(\xi_*^{-1}(\cdot, G_S))$ and $F_C(\cdot) = G_S(\eta_*^{-1}(\cdot, G_B))$. Because $\xi(\cdot, G_S)$ is uniquely determined by $G_S(\cdot)$ and $\eta(\cdot, G_B)$ is uniquely determined by $G_B(\cdot)$, it follows that $\xi_*(\cdot, G_S)$ and $\eta_*(\cdot, G_B)$ are uniquely determined by G . Hence, the private value distribution (F_V, F_C) that rationalizes G is unique. \square

B.4. Proof of Theorem 3. Given Theorem 2, it suffices to show that C5 and C6 are implied by C7 and C8.

We shall only show C7, more precisely, the monotonicity of $\xi(\cdot, G_S)$, implies C5. A similar argument can show that C8 implies C6. For buyer with private value v , since

$$\frac{\partial\pi_B(b, v)}{\partial b} = g_S(b) [v - \xi(b, G_S)],$$

then strictly increasing $\xi(\cdot, G_S)$ on $[\underline{s}, \bar{s}]$ ensures that for any $b \in (\xi^{-1}(v, G_S), \bar{s}]$, $\partial\pi_B(b, v)/\partial b < 0$, therefore, the expected profit of the buyer $\pi_B(b, v)$ is strictly decreasing in the buyer's bid. For $b' \in [\bar{b}, \bar{s}]$ and $b \leq \bar{b}$ such that $\xi(b, G_S) \geq b'$, let $v = \xi(b, G_S)$, then it follows from the above conclusion that

$$b' \geq \bar{b} \geq b = \xi^{-1}(v, G_S) \Rightarrow \pi_B(b', v) \leq \pi_B(b, v),$$

which is equivalent to C5 as shown in the proof of Theorem 1. \square

B.5. Proof of Lemma 2. By definition, G_{1B} and G_{1S} are the conditional marginal distributions of B and S given $(B, S) \in [\underline{s}, \bar{b}]^2$. So

$$\begin{aligned} G_{1B}(b) &\equiv \Pr(B \leq b \mid (B, S) \in [\underline{s}, \bar{b}]^2) = \int_{\underline{s}}^b \int_{\underline{s}}^{\bar{b}} g_1(x, y) \, dy \, dx, \\ G_{1S}(s) &\equiv \Pr(S \leq s \mid (B, S) \in [\underline{s}, \bar{b}]^2) = \int_{\underline{s}}^{\bar{b}} \int_{\underline{s}}^s g_1(x, y) \, dy \, dx, \\ g_{1B}(b) &\equiv \int_{\underline{s}}^{\bar{b}} g_1(b, y) \, dy, \quad g_{1S}(s) \equiv \int_{\underline{s}}^{\bar{b}} g_1(x, s) \, dx. \end{aligned}$$

By independence between buyer's offer and seller's ask, namely $g(b, s) = g_B(b)g_S(s)$, equation (3.9) implies that

$$1 - G_{1B}(b) = \int_b^{\bar{b}} \int_{\underline{s}}^{\bar{b}} \frac{1}{m} g_B(x) g_S(y) \, dy \, dx = \frac{1}{m} G_S(\bar{b}) [1 - G_B(b)], \quad (\text{B.10})$$

$$G_{1S}(s) = \int_{\underline{s}}^s \int_{\underline{s}}^{\bar{b}} \frac{1}{m} g_B(x) g_S(y) dx dy = \frac{1}{m} [1 - G_B(\underline{s})] G_S(s), \quad (\text{B.11})$$

$$g_{1B}(b) = \int_{\underline{s}}^{\bar{b}} \frac{1}{m} g_B(b) g_S(y) dy = \frac{1}{m} g_B(b) G_S(\bar{b}), \quad (\text{B.12})$$

$$g_{1S}(s) = \int_{\underline{s}}^{\bar{b}} \frac{1}{m} g_B(x) g_S(s) dx = \frac{1}{m} g_S(s) [1 - G_B(\underline{s})]. \quad (\text{B.13})$$

Thus, by (B.10)–(B.13),

$$\frac{G_S(b)}{g_S(b)} = \frac{G_{1S}(b)}{g_{1S}(b)}, \quad \frac{1 - G_B(s)}{g_B(s)} = \frac{1 - G_{1B}(s)}{g_{1B}(s)}.$$

So we finally get

$$\begin{aligned} \tilde{\xi}(b, G_{1S}) &= b + k \frac{G_{1S}(b)}{g_{1S}(b)} = b + k \frac{G_S(b)}{g_S(b)} = \xi(b, G_S), \\ \tilde{\eta}(s, G_{1B}) &= s - (1 - k) \frac{1 - G_{1B}(s)}{g_{1B}(s)} = s - (1 - k) \frac{1 - G_B(s)}{g_B(s)} = \eta(s, G_B) \end{aligned}$$

for all $b, s \in [\underline{s}, \bar{b}]$. □

B.6. Proof of Theorem 4. First, we show part (i). By Theorem 1, C1–C4 hold. By definition of G_2 , D1 is the direct corollary of C1. Using $g_2(b, s) = g(b, s)/m'$ and $g(b, s) = g_B(b)g_S(s)$ by C2, we have

$$g_2(b, s)g_2(b', s') = g_2(b, s')g_2(b', s) = \frac{g_B(b)g_B(b')g_S(s)g_S(s')}{m'^2},$$

so D2 holds. D3 is implied by C3 and (3.12); D4 is implied by C4 and (3.13).

Next, let us prove the conclusion of part (ii). Notice that, by D3 and D4, (3.14) is equivalent to

$$\frac{F_V(\tilde{\xi}(b, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(b), \quad \frac{F_C(\tilde{\eta}(s, G_{1B}))}{F_C(\bar{b})} = G_{1S}(s) \quad (\text{B.14})$$

for $(b, s) \in [\underline{s}, \bar{b}]^2$. For any (F_V, F_C) satisfying E1–E3, by Assumption B and D1, $\underline{c} \leq \underline{s} < \bar{b} \leq \bar{v}$. Consider the following strategy profile

$$\tilde{\beta}_B(v) = \begin{cases} v & \text{if } v \leq \underline{s}, \\ \tilde{\xi}^{-1}(v, G_{1S}) & \text{otherwise,} \end{cases} \quad (\text{B.15})$$

$$\tilde{\beta}_S(c) = \begin{cases} c & \text{if } c \geq \bar{b}, \\ \tilde{\eta}^{-1}(c, G_{1B}) & \text{otherwise,} \end{cases} \quad (\text{B.16})$$

where $\tilde{\xi}^{-1}(\cdot, G_{1S})$ and $\tilde{\eta}^{-1}(\cdot, G_{1B})$ are respective inverse functions of $\tilde{\xi}(\cdot, G_{1S})$ and $\tilde{\eta}(\cdot, G_{1B})$. D3 and D4 ensure that both $\tilde{\xi}^{-1}(\cdot, G_{1S})$ and $\tilde{\eta}^{-1}(\cdot, G_{1B})$ are well-defined, strictly increasing, and differentiable on $[\tilde{\xi}(\underline{s}, G_{1S}), \tilde{\xi}(\bar{b}, G_{1S})]$ and $[\tilde{\eta}(\underline{s}, G_{1B}), \tilde{\eta}(\bar{b}, G_{1B})]$, respectively.

We firstly show that $(\tilde{\beta}_B, \tilde{\beta}_S)$ will induce the same distribution of transacted bids as G_2 .

Since $\tilde{\xi}(\underline{s}, G_{1S}) = \underline{s}$ and $\tilde{\eta}(\bar{b}, G_{1B}) = \bar{b}$, so $\tilde{\xi}^{-1}(\underline{s}, G_{1S}) = \underline{s}$ and $\tilde{\eta}^{-1}(\bar{b}, G_{1B}) = \bar{b}$, then both $\tilde{\beta}_B$ and $\tilde{\beta}_S$ defined above are continuous and strictly increasing. Moreover, \bar{b} is the upper bound of G_{1B} 's

support, so by (B.14),

$$\frac{F_V(\tilde{\zeta}(\bar{b}, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(\bar{b}) = 1 \quad \Rightarrow \quad F_V(\tilde{\zeta}(\bar{b}, G_{1S})) = 1;$$

and for any $b < \bar{b}$,

$$\frac{F_V(\tilde{\zeta}(b, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(b) < 1 \quad \Rightarrow \quad F_V(\tilde{\zeta}(b, G_{1S})) < 1.$$

This means $\tilde{\zeta}(\bar{b}, G_{1S})$ should equal to the upper bound of F_V 's support, i.e. $\bar{v} = \tilde{\zeta}(\bar{b}, G_{1S})$. Similar argument can show that $\underline{c} = \tilde{\eta}(\underline{s}, G_{1B})$.

Define the induced bids $\tilde{B} = \tilde{\beta}_B(V)$ and $\tilde{S} = \tilde{\beta}_S(C)$. Then by the continuity and monotonicity of these strategies, we have the support of \tilde{B} is $[\tilde{\beta}_B(\underline{v}), \tilde{\beta}_B(\bar{v})] = [\underline{v}, \tilde{\zeta}^{-1}(\tilde{\zeta}(\bar{b}, G_{1S}), G_{1S})] = [\underline{v}, \bar{b}]$ and the support of \tilde{S} is $[\tilde{\beta}_S(\underline{c}), \tilde{\beta}_S(\bar{c})] = [\tilde{\eta}^{-1}(\tilde{\eta}(\underline{s}, G_{1B}), G_{1B}), \bar{c}] = [\underline{s}, \bar{c}]$. Since V and C are independent, so \tilde{B} and \tilde{S} are also independent. Thus for all $b, s \in [\underline{s}, \bar{b}]$,

$$\begin{aligned} \tilde{G}_{1B}(b) &\equiv \Pr(\tilde{B} \leq b \mid (\tilde{B}, \tilde{S}) \in [\underline{s}, \bar{b}]^2) = \Pr(\tilde{B} \leq b \mid \underline{s} \leq \tilde{B} \leq \bar{b}) \\ &= \Pr(\tilde{\beta}_B(V) \leq b \mid \underline{s} \leq \tilde{\beta}_B(V) \leq \bar{b}) = \Pr(V \leq \tilde{\zeta}(b, G_{1S}) \mid \tilde{\zeta}(\underline{s}, G_{1S}) \leq V \leq \tilde{\zeta}(\bar{b}, G_{1S})) \\ &= \frac{\Pr(\tilde{\zeta}(\underline{s}, G_{1S}) \leq V \leq \tilde{\zeta}(b, G_{1S}))}{\Pr(\tilde{\zeta}(\underline{s}, G_{1S}) \leq V \leq \tilde{\zeta}(\bar{b}, G_{1S}))} = \frac{\Pr(\underline{s} \leq V \leq \tilde{\zeta}(b, G_{1S}))}{\Pr(\underline{s} \leq V \leq \bar{v})} \\ &= \frac{F_V(\tilde{\zeta}(b, G_{1S})) - F_V(\underline{s})}{1 - F_V(\underline{s})} = G_{1B}(b), \\ \tilde{G}_{1S}(s) &\equiv \Pr(\tilde{S} \leq s \mid (\tilde{B}, \tilde{S}) \in [\underline{s}, \bar{b}]^2) = \Pr(\tilde{S} \leq s \mid \underline{s} \leq \tilde{S} \leq \bar{b}) \\ &= \Pr(\tilde{\beta}_S(C) \leq s \mid \underline{s} \leq \tilde{\beta}_S(C) \leq \bar{b}) = \Pr(C \leq \tilde{\eta}(s, G_{1B}) \mid \tilde{\eta}(\underline{s}, G_{1B}) \leq C \leq \tilde{\eta}(\bar{b}, G_{1B})) \\ &= \frac{\Pr(\tilde{\eta}(\underline{c}, G_{1B}) \leq C \leq \tilde{\eta}(s, G_{1B}))}{\Pr(\tilde{\eta}(\underline{s}, G_{1B}) \leq C \leq \tilde{\eta}(\bar{b}, G_{1B}))} = \frac{\Pr(\underline{c} \leq C \leq \tilde{\eta}(s, G_{1B}))}{\Pr(\underline{c} \leq C \leq \bar{c})} \\ &= \frac{F_C(\tilde{\eta}(s, G_{1B}))}{F_C(\bar{c})} = G_{1S}(s). \end{aligned}$$

Consequently, the corresponding conditional marginal density $\tilde{g}_{1B}(b) = g_{1B}(b)$ and $\tilde{g}_{1S}(s) = g_{1S}(s)$ for all $b, s \in [\underline{s}, \bar{b}]$.

By D2 and the definition of g_1 , we know that g_1 has the following property

$$\forall (b, s), (b', s') \in [\underline{s}, \bar{b}]^2 : \quad g_1(b, s)g_1(b', s') = g_1(b, s')g_1(b', s).$$

Then

$$\forall (b, s), (b', s') \in [\underline{s}, \bar{b}]^2 : \quad \frac{g_1(b', s')}{g_1(b, s')} = \frac{g_1(b', s)}{g_1(b, s)}.$$

Integrating both sides with respect to b' , we have

$$\frac{g_{1S}(s')}{g_1(b, s')} = \int_{\underline{s}}^{\bar{b}} \frac{g_1(b', s')}{g_1(b, s')} db' = \int_{\underline{s}}^{\bar{b}} \frac{g_1(b', s)}{g_1(b, s)} db' = \frac{g_{1S}(s)}{g_1(b, s)} \Rightarrow g_{1S}(s')g_1(b, s) = g_{1S}(s)g_1(b, s').$$

Integrating both sides again with respect to s' , we get

$$g_1(b, s) \int_{\underline{s}}^{\bar{b}} g_{1S}(s') ds' = g_1(b, s) = g_{1S}(s) \int_{\underline{s}}^{\bar{b}} g_1(b, s') ds' = g_{1B}(b) g_{1S}(s),$$

which holds for all $(b, s) \in [\underline{s}, \bar{b}]^2$. Thus, by the independence between \tilde{B} and \tilde{S} , the conditional joint density of (\tilde{B}, \tilde{S}) given $(\tilde{B}, \tilde{S}) \in [\underline{s}, \bar{b}]^2$, namely $\tilde{g}_1(b, s)$, satisfies

$$\tilde{g}_1(b, s) = \tilde{g}_{1B}(b) \tilde{g}_{1S}(s) = g_{1B}(b) g_{1S}(s) = g_1(b, s)$$

for all $(b, s) \in [\underline{s}, \bar{b}]^2$. Since

$$g_2(b, s) = \frac{g_1(b, s)}{\int_{\underline{s}}^{\bar{b}} \int_{\underline{s}}^s g_1(b, s) db ds}, \quad \tilde{g}_2(b, s) = \frac{\tilde{g}_1(b, s)}{\int_{\underline{s}}^{\bar{b}} \int_{\underline{s}}^s \tilde{g}_1(b, s) db ds},$$

thus the conditional density of the induced transacted bids (\tilde{B}, \tilde{S}) must be $\tilde{g}_2(b, s) = g_2(b, s)$ for all $(b, s) \in \mathcal{D}'$.

Secondly, we shall show that $(\tilde{\beta}_B, \tilde{\beta}_S)$ is a regular equilibrium for such (F_V, F_C) . Since A1–A3 in Assumption C are all satisfied by definitions of $\tilde{\beta}_B$ and $\tilde{\beta}_S$, it suffices to verify that $(\tilde{\beta}_B, \tilde{\beta}_S)$ maximizes the expected profit for buyer with $v > \underline{s}$ and seller with $c < \bar{b}$.

Let \tilde{G}_B and \tilde{G}_S denote the distributions of the induced bids $\tilde{B} = \tilde{\beta}_B(V)$ and $\tilde{S} = \tilde{\beta}_S(C)$, respectively. Then, for $v > \underline{s}$ and $b \in [\underline{s}, \bar{b}]$, by (2.1) we have

$$\frac{\partial \pi_B(b, v)}{\partial b} = -k \tilde{G}_S(b) + (v - b) \tilde{g}_S(b) = -k F_C(\tilde{\beta}_S^{-1}(b)) + (v - b) \frac{f_C(\tilde{\beta}_S^{-1}(b))}{\tilde{\beta}'_S(\tilde{\beta}_S^{-1}(b))}.$$

Since $\tilde{\beta}_S^{-1}(b) \leq \bar{b}$, then by (B.16) and (3.14),

$$F_C(\tilde{\beta}_S^{-1}(b)) = F_C(\bar{b}) G_{1S}(b), \quad f_C(\tilde{\beta}_S^{-1}(b)) = F_C(\bar{b}) g_{1S}(b) \tilde{\beta}'_S(\tilde{\beta}_S^{-1}(b)).$$

Hence,

$$\begin{aligned} \frac{\partial \pi_B(b, v)}{\partial b} &= -k F_C(\bar{b}) G_{1S}(b) + (v - b) F_C(\bar{b}) g_{1S}(b) \\ &= F_C(\bar{b}) g_{1S}(b) \left[v - \left(b + k \frac{G_{1S}(b)}{g_{1S}(b)} \right) \right] = F_C(\bar{b}) g_{1S}(b) [v - \tilde{\xi}(b, G_{1S})]. \end{aligned}$$

Because $F_C(\bar{b}) g_{1S}(b) > 0$, the monotonicity of $\tilde{\xi}(\cdot, G_{1S})$ by D3 implies that $b = \tilde{\xi}^{-1}(v, G_{1S}) = \tilde{\beta}_B(v)$ is the unique maximizer of the buyer's expected profit in $[\underline{s}, \bar{b}]$. For $b' \geq \bar{b}$ and $v \geq b'$, let $b = \tilde{\beta}_B(v)$, then by construction of $(\tilde{\beta}_B, \tilde{\beta}_S)$, we have $\tilde{G}_S(b') = F_C(b')$, $\tilde{G}_S(b) = F_C(\tilde{\beta}_S^{-1}(b)) = F_C(\tilde{\eta}(b, G_{1B}))$ and

$$\int_b^{b'} \tilde{G}_S(s) ds = \int_b^{\bar{b}} F_C(\tilde{\eta}(s, G_{1B})) ds + \int_{\bar{b}}^{b'} F_C(s) ds.$$

Note that by definition

$$\tilde{\xi}(b, \tilde{G}_S) = b + k \frac{\tilde{G}_S(b)}{\tilde{g}_S(b)} = b + k \frac{F_C(\tilde{\beta}_S^{-1}(b))}{f_C(\tilde{\beta}_S^{-1}(b)) / \tilde{\beta}'_S(\tilde{\beta}_S^{-1}(b))} = b + k \frac{G_{1S}(b)}{g_{1S}(b)} = \tilde{\xi}(b, G_{1S}),$$

therefore, (3.15) implies

$$[\check{\xi}(b, \tilde{G}_S) - b'] \tilde{G}_S(b') - [\check{\xi}(b, \tilde{G}_S) - b] \tilde{G}_S(b) + (1 - k) \int_b^{b'} \tilde{G}_S(s) ds \leq 0,$$

and it follows that $\pi_B(b', v) \leq \pi_B(b, v)$ under \tilde{G}_S . Hence, we show that the buyer would not bid any $b' \geq \bar{b}$ and $\tilde{\beta}_B$ gives the buyer the maximal expected profit. Similar argument can show the optimality of $\tilde{\beta}_S$ and it completes the proof of sufficiency of E1–E3.

It remains to be shown that only those (F_V, F_C) satisfying E1–E3 can rationalize the same distribution of transacted bids as G_2 .

By Lemma 1, we have already known that in a regular equilibrium the buyer will never bid higher than her private value and the seller will never bid lower than her private value, so the conditions $\underline{c} \leq \underline{s}$ and $\bar{v} \geq \bar{b}$ are straightforward. For any distribution of regular equilibrium bids, $G \in \mathcal{P}_{[b, \bar{b}] \times [s, \bar{s}]}$, such that G_2 is the corresponding distribution of transacted bids, the bids B and S are independent by Theorem 1, so G_1 that is uniquely derived from G_2 must be the corresponding conditional distribution of (B, S) given $(B, S) \in [s, \bar{b}]^2$. Since it has already be shown that $G_1(b, s) = G_{1B}(b)G_{1S}(s)$ because $g_1(b, s) = g_{1B}(b)g_{1S}(s)$, then G_{1B} is the conditional distribution of B given $B \geq \underline{s}$ and G_{1S} is the conditional distribution of S given $S \leq \bar{b}$, in other words,

$$G_{1B}(b) = \frac{G_B(b) - G_B(\underline{s})}{1 - G_B(\underline{s})}, \quad G_{1S}(s) = \frac{G_S(s)}{G_S(\bar{b})}. \quad (\text{B.17})$$

According to the proof of Theorem 2, G can only be rationalized by (F_V, F_C) defined in (B.6) and (B.7) which imply

$$F_V(\check{\xi}(b, G_S)) = G_B(b), \quad F_C(\eta(s, G_B)) = G_S(s) \quad (\text{B.18})$$

for $\underline{s} = \check{\xi}^{-1}(\underline{s}, G_S) \leq b \leq \bar{b}$ and $\underline{s} \leq s \leq \eta^{-1}(\bar{b}, G_B) = \bar{b}$. By (3.12), (3.13), (B.17), (B.18) and using $\check{\xi}(\underline{s}, G_{1S}) = \underline{s}$, $\tilde{\eta}(\bar{b}, G_{1B}) = \bar{b}$, we have condition (B.14) should hold for all (F_V, F_C) that can rationalize G_2 . In addition, according to Theorem 1, G is rationalizable only if G satisfies conditions C5 and C6. Given the equilibrium strategies are regular, we have $G_S(s) = F_C(s)$ for all $s > \bar{b}$ and $G_S(s) = F_C(\eta(s, G_B)) = F_C(\tilde{\eta}(s, G_{1B}))$ for all $s \leq \bar{b}$ by Lemma 2, therefore, (3.15) immediately follows from (3.7). A similar argument can show (3.16) follows from (3.8), too. The assertion of part (ii) is then established, which completes the proof. \square

B.7. Proof of Lemma 3. First, we will establish the following two properties on bidding strategies: (M1) under Assumption F, any regular equilibrium strategies β_B and β_S admit up to $R + 1$ continuous and bounded derivatives on $[s, \bar{v}]$ and $[c, \bar{b}]$, respectively; (M2) for any $v \in [s, \bar{v}]$ and any $c \in [c, \bar{b}]$, $\beta'_B(v) \geq \epsilon_B > 0$ and $\beta'_S(c) \geq \epsilon_S > 0$. To show (M1), we need to rewrite (3.1) and (3.2) as follows:

$$\beta'_S(c) = \frac{f_C(c) [\beta_B^{-1}(\beta_S(c)) - \beta_S(c)]}{k \cdot F_C(c)}, \quad (\text{B.19})$$

$$\beta'_B(v) = \frac{f_V(v) [\beta_B(v) - \beta_S^{-1}(\beta_B(v))]}{(1-k) \cdot [1 - F_V(v)]}. \quad (\text{B.20})$$

By definition, any pair of regular equilibrium strategies β_B and β_S is continuously differentiable on $[\underline{s}, \bar{v}]$ and $[\underline{c}, \bar{b}]$, respectively (see Assumption C). Consequently, under Assumption F, (B.19) and (B.20) imply that $\beta'_S(\cdot)$ and $\beta'_B(\cdot)$ are continuously differentiable on $[\underline{c}, \bar{b}]$ and $[\underline{s}, \bar{v}]$, respectively. This further implies that β_S and β_B are twice continuously differentiable on $[\underline{c}, \bar{b}]$ and $[\underline{s}, \bar{v}]$. Again, under Assumption F, (B.19) and (B.20) imply that $\beta'_S(\cdot)$ and $\beta'_B(\cdot)$ are twice continuously differentiable, and hence β_S and β_B admit up to third continuous bounded derivatives on $[\underline{c}, \bar{b}]$ and $[\underline{s}, \bar{v}]$, respectively. This argument can go on until we conclude that β_S and β_B admit up to $R + 1$ continuous bounded derivatives, respectively, on $[\underline{c}, \bar{b}]$ and $[\underline{s}, \bar{v}]$. This completes the proof of (M1).

Now we establish (M2). By definition of regular equilibrium, the seller's and buyer's bidding strategies are continuously differentiable with positive derivative on $[\underline{c}, \bar{b}]$ and $[\underline{s}, \bar{v}]$, respectively (see condition A2 of Assumption C), i.e., $\beta'_S(\cdot)$ and $\beta'_B(\cdot)$ are continuous and positive on $[\underline{c}, \bar{b}]$ and $[\underline{s}, \bar{v}]$. By extreme value theorem, $\beta'_S(\cdot)$ and $\beta'_B(\cdot)$ have positive minimum and maximum on $[\underline{c}, \bar{b}]$ and $[\underline{s}, \bar{v}]$, respectively. The conclusion of (M2) therefore follows.

It was shown earlier that $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ solve

$$\forall b, s \in [\underline{s}, \bar{v}] : \quad \beta_B(\xi(b, G_S)) = b, \quad \beta_S(\eta(s, G_B)) = s,$$

it follows from (M1), (M2) and Lemma C1 of [Guerre, Perrigne, and Vuong \(2000\)](#) that both $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ admit up to $R + 1$ continuous and bounded derivatives on $[\underline{s}, \bar{b}]$. Note that

$$g_B(b) = \frac{f_V(\beta_B^{-1}(b))}{\beta'_B(\beta_B^{-1}(b))}, \quad g_S(s) = \frac{f_C(\beta_S^{-1}(s))}{\beta'_S(\beta_S^{-1}(s))}.$$

In addition, f_V and f_C are bounded away from 0 by Assumption F, and β'_B and β'_S are bounded by (M2). The conclusion of part (i) then follows. Because $G_B(b) = F_V(\beta_B^{-1}(b)) = F_V(\xi(b, G_S))$ for $b \in [\underline{s}, \bar{b}]$, the result about G_B in part (ii) follows from that both $F_V(\cdot)$ and $\xi(\cdot, G_S)$ have $R + 1$ continuous and bounded derivatives on $[\underline{s}, \bar{b}]$. The result about G_S in part (ii) can be proven similarly. Lastly, to prove part (iii), we note that (3.3) and (3.4) give

$$g_S(s) = k \frac{G_S(s)}{\xi(s, G_S) - s}, \quad g_B(b) = (1-k) \frac{1 - G_B(b)}{b - \eta(b, G_B)}.$$

Since every term on the right-hand side admits up to $R + 1$ continuous and bounded derivatives, the desired result follows. \square

B.8. Proof of Lemma 4. We will first show part (ii), and then show part (i). For part (ii), we shall show the convergence rate of $\sup_{b \in \mathcal{C}_g} |\hat{g}_B(b) - g_B(b)|$, and the other conclusion can be proven analogously.

Note $\hat{\underline{s}} \geq \underline{s}$, $\hat{\bar{b}} \leq \bar{b}$ and as $n \rightarrow \infty$, $\hat{\underline{s}} \xrightarrow{p} \underline{s}$ and $\hat{\bar{b}} \xrightarrow{p} \bar{b}$. Given $\lim_{n \rightarrow \infty} h_B = 0$ by Assumption H1, for sufficiently large n , $\mathcal{C}_g \subset [\hat{\underline{s}} + h_B, \hat{\bar{b}} - h_B]$ and therefore the boundary-corrected kernel density estimator \hat{g}_B will be numerically identical to the standard kernel density estimator \tilde{g}_B . Thus, using

the existing results for the standard kernel density estimator (see Li and Racine (2006), page 31, Theorem 1.4), we have under Assumptions E to G and Assumption H1,

$$\sup_{b \in \mathcal{C}_g} |\hat{g}_B(b) - g_B(b)| = O_p \left(h_B^{R+1} + \sqrt{\frac{\log n}{nh_B}} \right) = O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R+1}{2R+3}} \right).$$

For part (i), since $\hat{s} \geq s$, $\hat{b} \leq \bar{b}$ and $|\hat{s} - s| = O_p(1/n)$, $|\hat{b} - \bar{b}| = O_p(1/n)$, the estimation error of \hat{s} and \hat{b} is negligible. Therefore the uniform consistency result on $[\underline{s}, \bar{b}]$ directly follows from the following lemma about the uniform convergence rate of our boundary-corrected kernel density estimator.²⁷

Lemma. *Suppose*

- (i) X_1, \dots, X_n are independently and identically distributed as F with density f and support $[\underline{x}, \bar{x}]$;
- (ii) f has r -th continuous bounded derivative on $[a, b] \subseteq [\underline{x}, \bar{x}]$ ($r = 1, 2$); $f(x) \geq c_0 > 0$ for all $x \in [a, b]$;
- (iii) The kernel K is symmetric with support $[-1, 1]$ and has twice continuous bounded derivative on \mathbb{R} , and K is of order 2, i.e. $\int_{-\infty}^{\infty} K(u) du = 1$, $\int_{-\infty}^{\infty} uK(u) du = 0$, $\int_{-\infty}^{\infty} u^2K(u) du = \kappa < \infty$;
- (iv) h satisfies $0 < h < (b - a)/2$, $h \rightarrow 0$ and $nh / \log n \rightarrow \infty$ as $n \rightarrow \infty$;
- (v) h' satisfies $h' = O(h)$ and $1/\sqrt{nh'^3} = O(h)$ as $n \rightarrow \infty$; $A > 1/3$.

Let \hat{f} be the boundary-corrected kernel density estimator on interval $[a, b] \subseteq [\underline{x}, \bar{x}]$ as defined in (4.1), then

$$\sup_{x \in [a, b]} |\hat{f}(x) - f(x)| = O_p \left(h^r + \sqrt{\frac{\log n}{nh}} \right).$$

Although g_B (or g_S) is discontinuous at \underline{s} (or \bar{b}), we can similarly use the boundary-corrected density kernel estimator to estimate g_B (or g_S) on interval $[\underline{b}, \underline{s}]$ (or interval $[\bar{b}, \bar{s}]$) and with the same argument we can get that \hat{g}_B (or \hat{g}_S) converges to the true density at the same rate as on interval $[\underline{s}, \bar{b}]$, then the desired uniform consistency results on the whole support of g_B or g_S follow. \square

B.9. Proof of Lemma 5. We first show part (ii). For part (ii), we shall show the convergence rate of $\sup_i \mathbb{1}(V_i \in \mathcal{C}_V) |\hat{V}_i - V_i|$. The other conclusion can be proven analogously.

Define $\mathcal{C}_B = \{b \in [\underline{s}, \bar{b}] \mid \zeta(b, G_S) \in \mathcal{C}_V\}$. Because $\zeta(\cdot, G_S)$ is a strictly increasing continuous function and \mathcal{C}_V is a closed inner subset of $[\underline{s}, \bar{v}]$, then \mathcal{C}_B is also a (fixed) closed inner subset of $[\underline{s}, \bar{b}]$. Hence, it follows from the definition of $\zeta(b, G_S)$ and (4.2) that

$$\begin{aligned} \mathbb{1}(V_i \in \mathcal{C}_V) |\hat{V}_i - V_i| &= \mathbb{1}(B_i \in \mathcal{C}_B) \cdot k \left| \frac{\hat{G}_S(B_i)}{\hat{g}_S(B_i)} - \frac{G_S(B_i)}{g_S(B_i)} \right| \\ &= \mathbb{1}(B_i \in \mathcal{C}_B) k \left| \frac{\hat{G}_S(B_i) - G_S(B_i)}{g_S(B_i)} - \frac{G_S(B_i)}{g_S(B_i)^2} [\hat{g}_S(B_i) - g_S(B_i)] \right. \\ &\quad \left. + o(\hat{G}_S(B_i) - G_S(B_i)) + o(\hat{g}_S(B_i) - g_S(B_i)) \right| \end{aligned}$$

²⁷ We omit the details in showing this lemma due to page limit. The full derivation is available upon request.

$$\begin{aligned}
&\leq \mathbf{1}(B_i \in \mathcal{C}_B) \left\{ \frac{|\hat{G}_S(B_i) - G_S(B_i)|}{g_S(B_i)} + \frac{G_S(B_i)}{g_S(B_i)^2} |\hat{g}_S(B_i) - g_S(B_i)| \right. \\
&\quad \left. + o(|\hat{G}_S(B_i) - G_S(B_i)|) + o(|\hat{g}_S(B_i) - g_S(B_i)|) \right\} \\
&\leq \sup_{B_i \in \mathcal{C}_B} \left\{ \frac{|\hat{G}_S(B_i) - G_S(B_i)|}{g_S(B_i)} + \frac{G_S(B_i)}{g_S(B_i)^2} |\hat{g}_S(B_i) - g_S(B_i)| \right. \\
&\quad \left. + o(|\hat{G}_S(B_i) - G_S(B_i)|) + o(|\hat{g}_S(B_i) - g_S(B_i)|) \right\} \\
&\leq \frac{\sup_{b \in \mathcal{C}_B} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2} \sup_{b \in \mathcal{C}_B} |\hat{g}_S(b) - g_S(b)| \\
&\quad + o\left(\sup_{b \in \mathcal{C}_B} |\hat{G}_S(b) - G_S(b)|\right) + o\left(\sup_{b \in \mathcal{C}_B} |\hat{g}_S(b) - g_S(b)|\right).
\end{aligned}$$

where the last inequality holds since, for any b , $g_S(b) \geq \alpha_S$ and $G_S(b) \leq 1$. Then,

$$\begin{aligned}
\sup_i \mathbf{1}(V_i \in \mathcal{C}_V) |\hat{V}_i - V_i| &\leq \frac{\sup_{b \in \mathcal{C}_B} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2} \sup_{b \in \mathcal{C}_B} |\hat{g}_S(b) - g_S(b)| \\
&\quad + o\left(\sup_{b \in \mathcal{C}_B} |\hat{G}_S(b) - G_S(b)|\right) + o\left(\sup_{b \in \mathcal{C}_B} |\hat{g}_S(b) - g_S(b)|\right).
\end{aligned}$$

Since $\sup_{b \in \mathcal{C}_B} |\hat{G}_S(b) - G_S(b)| \leq \sup_{b \in \mathbb{R}} |\hat{G}_S(b) - G_S(b)| = O_p(1/\sqrt{n})$, the desired result follows from Lemma 4-(ii) and $O_p\left(\max\left(1/\sqrt{n}, (\log n/n)^{(R+1)/(2R+3)}\right)\right) = O_p\left((\log n/n)^{(R+1)/(2R+3)}\right)$.

For part (i), by similar argument, we have

$$\begin{aligned}
\sup_i \mathbf{1}(V_i \in [\underline{s}, \bar{v}]) |\hat{V}_i - V_i| &\leq \frac{\sup_{b \in [\underline{s}, \bar{v}]} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2} \sup_{b \in [\underline{s}, \bar{v}]} |\hat{g}_S(b) - g_S(b)| \\
&\quad + o\left(\sup_{b \in [\underline{s}, \bar{v}]} |\hat{G}_S(b) - G_S(b)|\right) + o\left(\sup_{b \in [\underline{s}, \bar{v}]} |\hat{g}_S(b) - g_S(b)|\right)
\end{aligned}$$

Then it follows from Lemma 4-(i) that

$$\sup_i \mathbf{1}(V_i \in [\underline{s}, \bar{v}]) |\hat{V}_i - V_i| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right). \quad (\text{B.21})$$

Since by regular equilibrium assumption, the buyer with private value $v < \underline{s}$ will bid $b = v$ and hence $\hat{V}_i = B_i = V_i$. Then we can extend the result in (B.21) to all $V_i \in [\underline{v}, \bar{v}]$ so that

$$\sup_i |\hat{V}_i - V_i| = \sup_i \mathbf{1}(V_i \in [\underline{s}, \bar{v}]) |\hat{V}_i - V_i| = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right).$$

The result for $\sup_i |\hat{C}_i - C_i|$ can be shown analogously. \square

B.10. Proof of Theorem 5. We shall show the uniform consistency of $|\hat{f}_V(v) - f_V(v)|$, the other conclusion can be proven analogously.

First, we consider the case that \mathcal{C}_V is a closed inner subset of $[\underline{s}, \bar{v}]$. Let $\tilde{f}_V(v)$ define the (infeasible) one-step boundary-corrected kernel density estimator which uses the unobserved true private values V_i instead of \hat{V}_i . Applying similar argument to show Lemma 4, we can show that $\sup_{v \in \mathcal{C}_V} |\tilde{f}_V(v) - f_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$ given non-optimal bandwidth $h_V = \lambda_V(\log n/n)^{1/(2R+3)}$. Since $\hat{f}_V(v) - f_V(v) = [\hat{f}_V(v) - \tilde{f}_V(v)] + [\tilde{f}_V(v) - f_V(v)]$, we are left with the first term.

Let $\mathcal{C}'_V = \cup_{v \in \mathcal{C}_V} [v - \Delta, v + \Delta]$ and $\mathcal{C}''_V = \cup_{v \in \mathcal{C}'_V} [v - \Delta, v + \Delta]$ for some $\Delta > 0$. By construction, \mathcal{C}'_V and \mathcal{C}''_V are also closed, and $\mathcal{C}_V \subset \mathcal{C}'_V \subset \mathcal{C}''_V$. Since \mathcal{C}_V is a closed inner subset of $[\underline{s}, \bar{v}]$, Δ can be chosen small enough such that $\mathcal{C}''_V \subset [\underline{s}, \bar{v}]$. Now by Lemma 5, for $v \in \mathcal{C}_V$ and n large enough, $\hat{f}_V(v)$ uses at most observations \hat{V}_i in \mathcal{C}'_V and for which V_i is in \mathcal{C}''_V . Because for any $v \in \mathcal{C}_V$, $\tilde{f}_V(v)$ uses at most V_i in \mathcal{C}''_V and both $\hat{f}_V(v)$ and $\tilde{f}_V(v)$ are numerically identical to the standard kernel density estimator, we obtain

$$\hat{f}_V(v) - \tilde{f}_V(v) = \frac{1}{nh_V} \sum_{i=1}^n \mathbf{1}(V_i \in \mathcal{C}''_V) \left[K_V \left(\frac{v - \hat{V}_i}{h_V} \right) - K_V \left(\frac{v - V_i}{h_V} \right) \right].$$

A second-order Taylor expansion gives

$$\begin{aligned} |\hat{f}_V(v) - \tilde{f}_V(v)| &= \left| \frac{1}{nh_V} \sum_{i=1}^n \left[\mathbf{1}(V_i \in \mathcal{C}''_V) (\hat{V}_i - V_i) \cdot \frac{1}{h_V} K'_V \left(\frac{v - V_i}{h_V} \right) \right] \right. \\ &\quad \left. + \frac{1}{2nh_V} \sum_{i=1}^n \left[\mathbf{1}(V_i \in \mathcal{C}''_V) (\hat{V}_i - V_i)^2 \cdot \frac{1}{h_V^2} K''_V \left(\frac{v - \tilde{V}_i}{h_V} \right) \right] \right| \end{aligned}$$

where \tilde{V}_i is some point between \hat{V}_i and V_i . By triangular inequality,

$$\begin{aligned} |\hat{f}_V(v) - \tilde{f}_V(v)| &\leq \frac{1}{nh_V^2} \sum_{i=1}^n \mathbf{1}(V_i \in \mathcal{C}''_V) |\hat{V}_i - V_i| \cdot \left| K'_V \left(\frac{v - V_i}{h_V} \right) \right| \\ &\quad + \frac{1}{2nh_V^3} \sum_{i=1}^n \mathbf{1}(V_i \in \mathcal{C}''_V) (\hat{V}_i - V_i)^2 \cdot \left| K''_V \left(\frac{v - \tilde{V}_i}{h_V} \right) \right|. \quad (\text{B.22}) \end{aligned}$$

Because $\left| K''_V \left(\frac{v - \tilde{V}_i}{h_V} \right) \right| \leq \sup_u |K''_V(u)|$, then the right-hand side of (B.22) is bounded by

$$\frac{1}{h_V} \sup_i \mathbf{1}(V_i \in \mathcal{C}''_V) |\hat{V}_i - V_i| \cdot \frac{1}{nh_V} \sum_{i=1}^n \left| K'_V \left(\frac{v - V_i}{h_V} \right) \right| + \frac{1}{2h_V^3} \sup_i \mathbf{1}(V_i \in \mathcal{C}''_V) |\hat{V}_i - V_i|^2 \cdot \sup_u |K''_V(u)|.$$

By Lemma 5-(ii) and Assumption H1,

$$\begin{aligned} |\hat{f}_V(v) - \tilde{f}_V(v)| &\leq O_p \left(\left(\frac{\log n}{n} \right)^{\frac{R}{2R+3}} \right) \cdot \frac{1}{nh_V} \sum_{i=1}^n \left| K'_V \left(\frac{v - V_i}{h_V} \right) \right| + O_p \left(\left(\frac{\log n}{n} \right)^{\frac{2R-1}{2R+3}} \right) \cdot \sup_u |K''_V(u)|. \end{aligned} \quad (\text{B.23})$$

It can be shown that $\frac{1}{nh_V} \sum_{i=1}^n \left| K'_V \left(\frac{v - V_i}{h_V} \right) \right|$ converges uniformly to $f_V(v) \int_{-\infty}^{\infty} |K'_V(u)| du$ thus it is bounded uniformly. Moreover, $\sup_u |K''_V(u)| < \infty$ by Assumption G. Since $R \geq 1$ implies

$\frac{2R-1}{2R+3} \geq \frac{R}{2R+3}$, it follows that $\sup_{v \in \mathcal{C}_V} |\hat{f}_V(v) - \tilde{f}_V(v)| = O_p \left((\log n/n)^{R/(2R+3)} \right)$ and therefore $\sup_{v \in \mathcal{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p \left((\log n/n)^{R/(2R+3)} \right)$.

Now we consider the other case that \mathcal{C}_V is a closed inner subset of $[\underline{v}, \underline{s}]$ when $\underline{s} > \underline{v}$. By regular equilibrium assumption, the buyer with private value $v < \underline{s}$ will bid $b = v$, thus we have $\hat{V}_i = B_i = V_i$. Thus \hat{f}_V is in fact the one-step boundary-corrected kernel estimator for f_V on $[\underline{v}, \underline{s}]$. Same argument gives $\sup_{v \in \mathcal{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p \left((\log n/n)^{R/(2R+3)} \right)$.

Since any given closed inner subset $\mathcal{C}_V \subseteq [\underline{v}, \bar{v}] \setminus \{\underline{s}\}$ is a union of at most two closed inner subsets respectively belonging to the two cases above, the final conclusion is proven. \square

B.11. Proof of Theorem 6. This conclusion is obtained by applying a similar argument to that in Theorem 5 where we show the uniform convergence rate of $\hat{f}_V(\cdot)$ (or $\hat{f}_C(\cdot)$) in the closed inner subset of $[\underline{s}, \bar{v}]$ (or $[\underline{c}, \bar{b}]$). However, we use part (i) (instead of part (ii)) of Lemma 5 here. \square

B.12. Proof of Lemma 6. First, note that when $k \in (0, 1]$, we can rewrite (A.1) and (A.3) together as

$$\Psi_k(p) = \int_{\underline{s}}^p \int_s^{\min\left(\frac{p-(1-k)s}{k}, \bar{b}\right)} g_2(b, s) db ds. \quad (\text{B.24})$$

Keep $p \in (\underline{s}, \bar{b})$ fixed and define a function φ as the inner integral in (B.24), i.e.

$$\varphi(k, s) = \int_s^{\min\left(\frac{p-(1-k)s}{k}, \bar{b}\right)} g_2(b, s) db, \quad k \in (0, 1], s \in [\underline{s}, p]. \quad (\text{B.25})$$

Since $g_2(b, s)$ is integrable, so φ is continuous in the upper limit of integral. And since the upper limit, $\min\left(\frac{p-(1-k)s}{k}, \bar{b}\right)$, is continuous in k , so φ is continuous in k . Note that $g_2(b, s) > 0$ because the interval of integration is in the support of G , and note that $\min\left(\frac{p-(1-k)s}{k}, \bar{b}\right) \leq \bar{b}$, thus for any $k \in (0, 1]$,

$$0 \leq \varphi(k, s) \leq \int_s^{\bar{b}} g_2(b, s) db \equiv \bar{\varphi}(s), \quad \forall s \in [\underline{s}, p].$$

Therefore, for any $k \in (0, 1]$, for any sequence $\{k_n\}$ in $(0, 1]$ such that $k_n \rightarrow k$ as $n \rightarrow \infty$, by continuity of φ in k , we have $\bar{\varphi}_n(s) \equiv \varphi(k_n, s)$ converges pointwise to $\bar{\varphi}(s) \equiv \varphi(k, s)$ on $[\underline{s}, p]$. Since $\bar{\varphi}(s)$ is integrable, by dominated convergence theorem, as $n \rightarrow \infty$,

$$\int_{\underline{s}}^p \bar{\varphi}_n(s) ds \rightarrow \int_{\underline{s}}^p \bar{\varphi}(s) ds,$$

hence, $\Psi_{k_n}(p) \rightarrow \Psi_k(p)$.

To see the (right) continuity at $k = 0$, we just need to rewrite (A.1) and (A.2) as

$$\Psi_k(p) = 1 - \int_p^{\bar{b}} \int_{\frac{p-kb}{1-k}}^b g_2(b, s) ds db, \quad 0 \leq k < \frac{p - \underline{s}}{b - \underline{s}}$$

and define

$$\psi(k, b) = - \int_{\frac{p-kb}{1-k}}^b g(b, s) ds, \quad k \in \left[0, \frac{p - \underline{s}}{b - \underline{s}}\right), b \in [p, \bar{b}].$$

Then applying analogous argument, we have ψ is continuous in k so that for sequence $\{k_n\}$ in $\left[0, \frac{p-s}{b-s}\right)$ such that $k_n \rightarrow 0$, the sequence $\{\tilde{\psi}_n(b) \equiv \psi(k_n, b)\}$ converges pointwise to $\tilde{\psi}(b) \equiv \psi(0, b)$. Since $\{\tilde{\psi}_n(b)\}$ is dominated by $\tilde{\psi}(b) \equiv \int_{\underline{s}}^b g(b, s) ds$, we finally can get $\Psi_{k_n}(p) \rightarrow \Psi_0(p)$.

It remains to show the monotonicity of $\Psi_k(p)$ in k . Suppose $0 \leq k_1 < k_2 \leq 1$, then by (A.1), (A.2), and (A.3):

(i) If $k_2 < \frac{p-s}{b-s}$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_p^{\bar{b}} \int_{b-\frac{b-p}{1-k_2}}^{b-\frac{b-p}{1-k_1}} g_2(b, s) ds db > 0$$

due to $\frac{b-p}{1-k_2} > \frac{b-p}{1-k_1}$.

(ii) If $k_1 \geq \frac{p-s}{b-s}$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{\underline{s}}^p \int_{s+\frac{p-s}{k_2}}^{s+\frac{p-s}{k_1}} g_2(b, s) db ds > 0$$

due to $\frac{p-s}{k_2} < \frac{p-s}{k_1}$.

(iii) If $k_1 < \frac{p-s}{b-s} \leq k_2$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{\underline{s}}^p \int_{s+\frac{p-s}{k_2}}^{s+\frac{(p-s)(\bar{b}-\underline{s})}{p-s}} g_2(b, s) db ds + \int_p^{\bar{b}} \int_{b-\frac{(b-p)(\bar{b}-\underline{s})}{b-p}}^{b-\frac{b-p}{1-k_1}} g_2(b, s) ds db > 0,$$

where the first term is non-negative and the second one is positive. \square